

POINTS WITH FINITE ORBITS FOR TRACE MAPS

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ABSTRACT. We study an action of $\text{Aut}(F_n)$ on \mathbb{R}^{2^n-1} by trace maps, defined using the traces of n -tuples of matrices in $\text{SL}(2, \mathbb{C})$ having real traces. We determine the finite orbits for this action. These orbits essentially come from (i) the finite subgroups of $\text{SL}(2, \mathbb{C})$, and (ii) a dense set of (rational) points in an embedded quotient of an n -torus.

0. INTRODUCTION

Many authors have studied trace maps [1, 5, 7, 16, 17, 19, 22, 23, 24, 26], which give an action of $\text{Aut}(F_2)$ on \mathbb{R}^3 (here F_n is a rank n free group) and form an interaction between representation theory and dynamical systems. Goldman and others have also studied this action in the context of character varieties, see [10, 11, 12, 3, 4, 27]. The case $n = 2$ is that usually studied. Here there are elements $\sigma_1, \sigma_2 \in \text{Aut}(F_2)$ acting as follows on \mathbb{R}^3 :

$$(0.1) \quad \sigma_1(x, y, z) = (z, y, 2yz - x), \quad \sigma_2(x, y, z) = (x, 2xy - z, y).$$

This comes from an action of $\text{Aut}(F_2)$ on triples (x_1, x_2, x_{12}) of traces corresponding to pairs of matrices $A_1, A_2 \in \text{SL}(2, \mathbb{C})$, where

$$(0.2) \quad x_1 = \text{trace}(A_1)/2, \quad x_2 = \text{trace}(A_2)/2, \quad x_{12} = \text{trace}(A_1 A_2)/2,$$

are real numbers; see §1 for more details.

The $\text{Aut}(F_2)$ action preserves the level surfaces $E_t = E^{-1}(t)$ of the function

$$(0.3) \quad E : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad E(x, y, z) = x^2 + y^2 + z^2 - 2xyz.$$

A point of E_t will be said to *lie on level t* . Now E_1 contains (within the cube $[-1, 1]^3$) a curvilinear tetrahedron that is parameterised as:

$$(0.4) \quad \partial\mathcal{T} = \{(\cos 2\pi\theta_1, \cos 2\pi\theta_2, \cos 2\pi(\theta_1 + \theta_2)) : (\theta_1, \theta_2)^T \in \mathbb{R}^2\},$$

where the action of $\text{Aut}(F_2)$ is given via the corresponding $\text{GL}(2, \mathbb{Z})$ -action on $(\theta_1, \theta_2)^T$.

Thus, for $m \in \mathbb{N}$, any $\sigma \in \text{Aut}(F_2)$ permutes the points of $\partial\mathcal{T}$ corresponding to $\{(p/m, q/m)^T : 0 \leq p, q < m\}$, which therefore have finite $\text{Aut}(F_2)$ -orbits.

One can generalize the above to the situation where, for $n \geq 2$, the automorphism group $\text{Aut}(F_n)$ acts on a trace variety corresponding to n -tuples of matrices

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$A_1, A_2, \dots, A_n \in \mathrm{SL}(2, \mathbb{C})$ with real traces. In this case the trace variety is generated by the $2^n - 1$ half-traces

$$x_I = \mathrm{trace}(A_{i_1} A_{i_2} \dots A_{i_k})/2, \quad I = (i_1, i_2, \dots, i_k), \\ 1 \leq k \leq n, 1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

Here ‘generated’ means that for any $A \in \langle A_1, A_2, \dots, A_n \rangle$, the half-trace of A is an integer polynomial in these x_I . We note that these $2^n - 1$ traces are certainly not a minimal generating set (see [20, 9] for example), however they are natural in our situation.

Thus we obtain an action of $\mathrm{Aut}(F_n)$ on \mathbb{R}^{2^n-1} .

The aim of this paper is to study the set $\mathcal{F}_n, n = 2, 3$, of points of \mathbb{R}^{2^n-1} which have finite orbit under the action of $\mathrm{Aut}(F_n)$ and the set \mathcal{P}_n of points which are periodic for each element of $\mathrm{Aut}(F_n)$. It is clear that $\mathcal{F}_n \subseteq \mathcal{P}_n$ and the main result of this paper is to prove that $\mathcal{F}_n = \mathcal{P}_n$ and to completely determine \mathcal{F}_n .

The result in the case $n = 2$ follows from results of [8], where the authors use this result to study the global analytic properties of the solutions of a particular family of Painlevé VI equations. Here we note that for any $(x, y, z)^T \in \mathbb{R}^3 \setminus E_1$ there is a pair $(A_1, A_2) \in \mathrm{SL}(2, \mathbb{Z})$ (given explicitly in §1), determined up to conjugacy, and which we say is *associated to* $(x, y, z)^T$. The result of [8] is:

Theorem 1. *Consider the action of $\mathrm{Aut}(F_2)$ on \mathbb{R}^3 . Let $\mathcal{F} = \mathcal{F}_2$ denote the points that have finite $\mathrm{Aut}(F_2)$ -orbit and $\mathcal{P} = \mathcal{P}_2$ those that have finite orbit under each element of $\mathrm{Aut}(F_2)$. Then any point $p \in \mathcal{P}$ with associated matrices (A_1, A_2) satisfies one of the following three conditions:*

- (1) *the group $\langle A_1, A_2 \rangle$ is a finite group;*
- (2) *p lies on one of the coordinate axes of \mathbb{R}^3 ;*
- (3) *the pair (A_1, A_2) is conjugate to (A'_1, A'_2) where A'_1, A'_2 are lower triangular and where the diagonal elements of A'_1 and A'_2 are roots of unity. This includes the case where some word in $\langle A_1, A_2 \rangle$ is parabolic (so that the group $\langle A_1, A_2 \rangle$ is not finite).*

Moreover $\mathcal{P} = \mathcal{F}$.

It is easy to see that $p \in \mathcal{F}$ in each of the above three cases.

For (1) in the above Theorem we recall the well known fact that any finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ is cyclic, binary dihedral or a subgroup of one of:

The binary tetrahedral group

$$\mathrm{BT}_{24} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+i & i-1 \\ 1+i & 1-i \end{pmatrix} \right\rangle.$$

Note that $\mathrm{BT}_{24} \cong \mathrm{SL}(2, 3)$.

The binary octahedral group

$$\mathrm{BO}_{48} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+i & i-1 \\ 1+i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \right\rangle.$$

The binary icosahedral group

$$\mathrm{BI}_{120} = \left\langle \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon^4 - \varepsilon & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix} \right\rangle,$$

where ε is a primitive fifth root of unity.

From the classification of Theorem 1 it follows that if $p \notin E_1$ and p does not lie on an axis, then the associated group $\langle A_1, A_2 \rangle$ is finite, and p is in one of $\text{Aut}(F_2)$ -orbits O_1, \dots, O_5 of sizes 40, 36, 72, 16, 40 (respectively).

The fact that $\text{Inn}(F_2)$ acts trivially on the triples of traces (x_1, x_2, x_{12}) means that we are really considering an action of $\text{Out}(F_2) = \text{Aut}(F_2)/\text{Inn}(F_2)$.

We should note that in [8] the authors consider the action of the braid group B_3 (instead of $\text{Aut}(F_2)$) on such triples. We further note however that this action of B_3 is determined by the elements $\sigma_1, \sigma_2 \in \text{Aut}(F_2)$ (defined in the next section). It is a fact that the subgroup $\langle \sigma_1, \sigma_2 \rangle \cong \text{PSL}(2, \mathbb{Z}) \cong B_3/Z(B_3)$ has finite index (8) in $\text{Out}(F_2)$. Thus the problem of determining points of \mathbb{R}^3 with finite orbit for the action of $\text{Aut}(F_2)$ is the same as the problem of finding points with finite orbit for the action of the subgroup $\langle \sigma_1, \sigma_2 \rangle$.

We also note that the authors of [8] are only interested in the triples $(x_1, x_2, x_{12}) \in \mathbb{R}^3$ (with finite orbit) up to an equivalence where two such are equivalent if they differ by changing the sign of two of the entries. Thus they obtain orbits of sizes 10, 9, 18, 4, 10 (respectively).

We also note that in [8] they are only concerned with points having finite orbits that are in the interior of \mathcal{T} . Results given in §3, together with what is proved in [8], easily give Theorem 1. The last thing to note about [8] is that they use the traces of the products $A_{i_1}A_{i_2} \dots A_{i_k}$, not the half-traces.

In the next few sections we introduce the preliminary results for the $n = 2, 3$ cases, finally being able to state our main result for the case $n = 3$ at the end of section 2. This result is proved in §§3 – 7. The result for general n is proved in §8.

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1. PRELIMINARIES FOR $n = 2$

Let $F_2 = \langle a_1, a_2 \rangle$ be a free group of rank 2 and let $\sigma_i \in \text{Aut}(F_2), i = 1, 2$, be defined by

$$\begin{aligned} \sigma_1(a_1) &= a_1a_2, & \sigma_1(a_2) &= a_2; \\ \sigma_2(a_1) &= a_1, & \sigma_2(a_2) &= a_1^{-1}a_2. \end{aligned}$$

One can show that σ_1, σ_2 satisfy the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and that $(\sigma_1\sigma_2)^3$ acts as an inner automorphism, so that the action of $(\sigma_1\sigma_2)^3$ on the trace triples is trivial. Under the natural homomorphism

$$\Phi = \Phi_2 : \text{Aut}(F_2) \rightarrow \text{GL}(2, \mathbb{Z}),$$

we have

$$\Phi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Phi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Note [21, Th 3.9] that any element of $\text{Aut}(F_2)$ fixes the commutator $a_1a_2a_1^{-1}a_2^{-1}$ up to conjugacy and inversion.

Now suppose that the $a_i, i = 1, 2$, are represented by elements $A_1, A_2 \in \text{SL}(2, \mathbb{C})$. Define x_1, x_2, x_{12} as in (0.2). Recall the standard trace identities for such 2×2

matrices:

$$\begin{aligned} \text{trace}(A_1^{-1}) &= \text{trace}(A_1), & \text{trace}(I_2) &= 2, \\ \text{trace}(A_1 A_2) &= \text{trace}(A_2 A_1) = \text{trace}(A_1) \text{trace}(A_2) - \text{trace}(A_1 A_2^{-1}). \end{aligned}$$

Using these we obtain the induced action of σ_1, σ_2 on \mathbb{R}^3 given by (0.1).

We will write this action of $\alpha \in \text{Aut}(F_2)$ on $(x, y, z)^T \in \mathbb{R}^3$ on the right: $(x, y, z)^T \alpha$; this action is also the corresponding action by Nielsen transformations [21, 25]. In [16, 17] we studied the family of trace maps $\sigma_1^n \sigma_2^n$ and determined all of their curves of fixed points and some of their period 2 curves.

One can check that for all $A_1, A_2 \in \text{SL}(2, \mathbb{C})$ we have $\text{trace}(A_1 A_2 A_1^{-1} A_2^{-1})/2 = 2E(x_1, x_2, x_{12}) - 1$. Thus, from the above trace identities and the fact that any element of $\text{Aut}(F_2)$ fixes the commutator $a_1 a_2 a_1^{-1} a_2^{-1}$ up to conjugacy and inversion, it follows that the action of $\text{Aut}(F_2)$ fixes the function $E = E(x_1, x_2, x_{12})$ of (0.3). Thus each level set E_t is invariant under the action. The level set E_1 is distinguished and has been drawn by many authors [10, 24]. There are four points,

$$V = \{(1, 1, 1)^T, (-1, -1, 1)^T, (-1, 1, -1)^T, (1, -1, -1)^T\},$$

in E_1 which are the only singular points of E_1 . In fact they are the only singular points of any E_t . Further, the six line segments joining these points are contained in E_1 and there is a unique component of $E_1 \setminus V$ whose closure is compact. In fact this closure is a topological 2-sphere that separates \mathbb{R}^3 into two components, the closure of one of these components is a 3-ball \mathcal{T} that we call a “curvilinear tetrahedron”, for whose boundary we gave a parametrisation in (0.4). One can check that $\mathcal{T} \subseteq [-1, 1]^3$ and that $\mathcal{T} \cap \partial[-1, 1]^3$ is the above mentioned set of six line segments. Further the closures of the other (four) components of $E_1 \setminus V$ are determined by the point of V which they contain; we will call these components *cones*.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the 2-torus. Then we have the map

$$\Pi_2 : \mathbb{T}^2 \rightarrow \partial\mathcal{T}, \quad (\theta_1, \theta_2)^T \mapsto (\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1 + \theta_2)))^T.$$

Note that $\Pi_2(\theta_1, \theta_2)^T = \Pi_2(-(\theta_1, \theta_2)^T)$. Then the map Π_2 is a branched double cover, branched over the four points of V .

The action of $\text{Aut}(F_2)$ on $\partial\mathcal{T}$ actually comes from the action of $\text{GL}(2, \mathbb{Z})$ on \mathbb{T}^2 , the action being determined by the homomorphism Φ_2 . For $\alpha \in \text{Aut}(F_2)$ and $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \in \mathbb{T}^2$ the maps Π_2, Φ_2 are related as follows (see [16, p. 1170]):

$$(1.1) \quad (\Pi_2 \theta) \alpha = \Pi_2(\Phi_2(\alpha)(\theta)).$$

The set $\mathcal{F} = \mathcal{F}_2$ includes the points V . If we ignore the points of V for the moment, then, as pointed out in [24, p. 839], a consequence of the implicit function theorem is that, for any point $p \in \mathcal{P} \cap \partial\mathcal{T}$ and $\sigma \in \text{Aut}(F_2)$ with $\Phi_2(\sigma)$ hyperbolic, there is a curve of fixed points of σ^N through p for some $N = N(p, \sigma)$; see also [3, §5].

To every point $(x_1, x_2, x_{12})^T \in \mathbb{R}^3$ there is a pair $(A_1, A_2) \in \text{SL}(2, \mathbb{C})^2$ such that $\text{trace}(A_1) = X = 2x_1$, $\text{trace}(A_2) = Y = 2x_2$ and $\text{trace}(A_1 A_2) = Z = 2x_{12}$. The pair (A_1, A_2) is determined up to conjugacy by the point $(x_1, x_2, x_{12})^T \notin E_1$ and

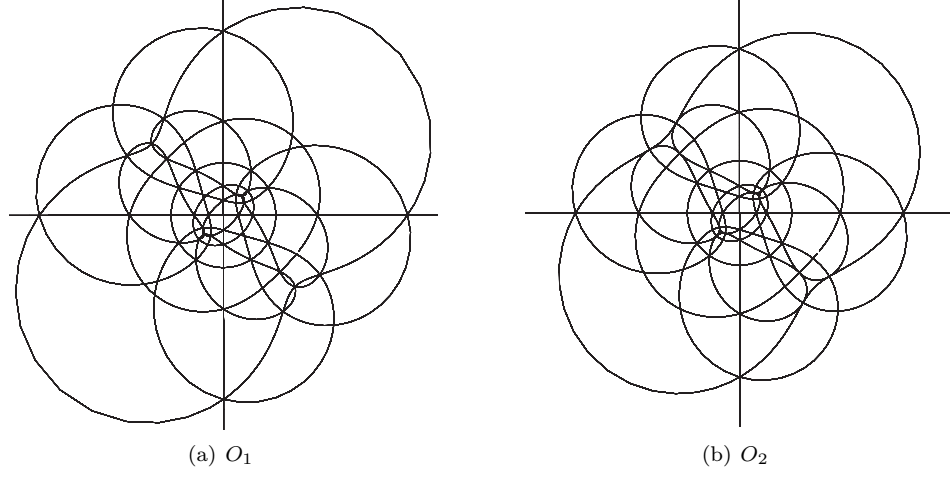


FIGURE 1.

one possible choice for A_1, A_2 is

$$A_1 = \begin{pmatrix} \frac{X+\sqrt{X^2-4}}{2} & 0 \\ 0 & \frac{2}{X+\sqrt{X^2-4}} \end{pmatrix},$$

$$(1.2) \quad A_2 = \begin{pmatrix} \frac{XZ+Z\sqrt{X^2-4}-2Y}{X^2-4+X\sqrt{X^2-4}} & \frac{X^2+Y^2+Z^2-XYZ-4}{4-X^2} \\ 1 & \frac{-XZ-Z\sqrt{X^2-4}-2Y+X^2Y+XY\sqrt{X^2-4}}{X^2-4+X\sqrt{X^2-4}} \end{pmatrix}.$$

We note that $(x_1, x_2, x_{12}) \in E_1$ if and only if the $(1, 2)$ entry of the above A_2 is zero, so that A_1, A_2 are both lower triangular matrices if $(x_1, x_2, x_{12}) \in E_1$.

It follows from the proof of Theorem 1 in [8] that there are only five finite $\text{Aut}(F_2)$ orbits which do not lie in E_1 or on the axes. We denote them by O_1, \dots, O_5 and they contain (respectively) 40, 36, 72, 16, 40 points. They lie on levels

$$0.904508\dots, \quad 3/4, \quad 3/4, \quad 1/2, \quad 0.3454915\dots$$

and each has associated group $\langle A_1, A_2 \rangle$ a finite group of order 120, 48, 120, 24, 120 (respectively), the group of order 120 being BI_{120} . (See [18, Th 29.6] for the possible finite subgroups of $SU(2, \mathbb{C})$.) Thus there are four levels that contain the five sets $O_i, i = 1, \dots, 5$.

In order to give the reader an idea of what the points of each O_i look like we draw the following diagrams. For every point $p = (x, y, z)^T \in O_i$, we take the simple closed curves containing p which are parallel to the various coordinate planes and which lie on the same level as p ; there will be three such curves through every $p \in O_i$. We then radially project these curves onto the unit sphere and then stereographically project onto the xy -plane. These curves are drawn in Figures 1, 2. The points of O_i in the Figures are exactly those points which are triple points for these curves. We further note that each such simple closed curve contains many of the points of O_i , all such points in a single curve being in one or more orbits of a conjugate of σ_1 .

From the above one easily deduces:

Corollary 1.1. *The half-trace values for the points in the orbits O_i , $1 \leq i \leq 5$, are*

$$0, \pm 1/2, \pm 1/\sqrt{2}, (\pm 1 \pm \sqrt{5})/4. \quad \square$$

We now list further properties of the orbits O_1, \dots, O_5 .

72 There is a single $\text{Aut}(F_2)$ orbit O_3 of size 72. This orbit is on level $3/4$. The permutation action of $\text{Aut}(F_2)$ on O_3 gives a group of order $2^{34} \cdot 3^5 \cdot 5 \cdot 7$ which is an imprimitive permutation group: the 72 points can be partitioned into 9 blocks

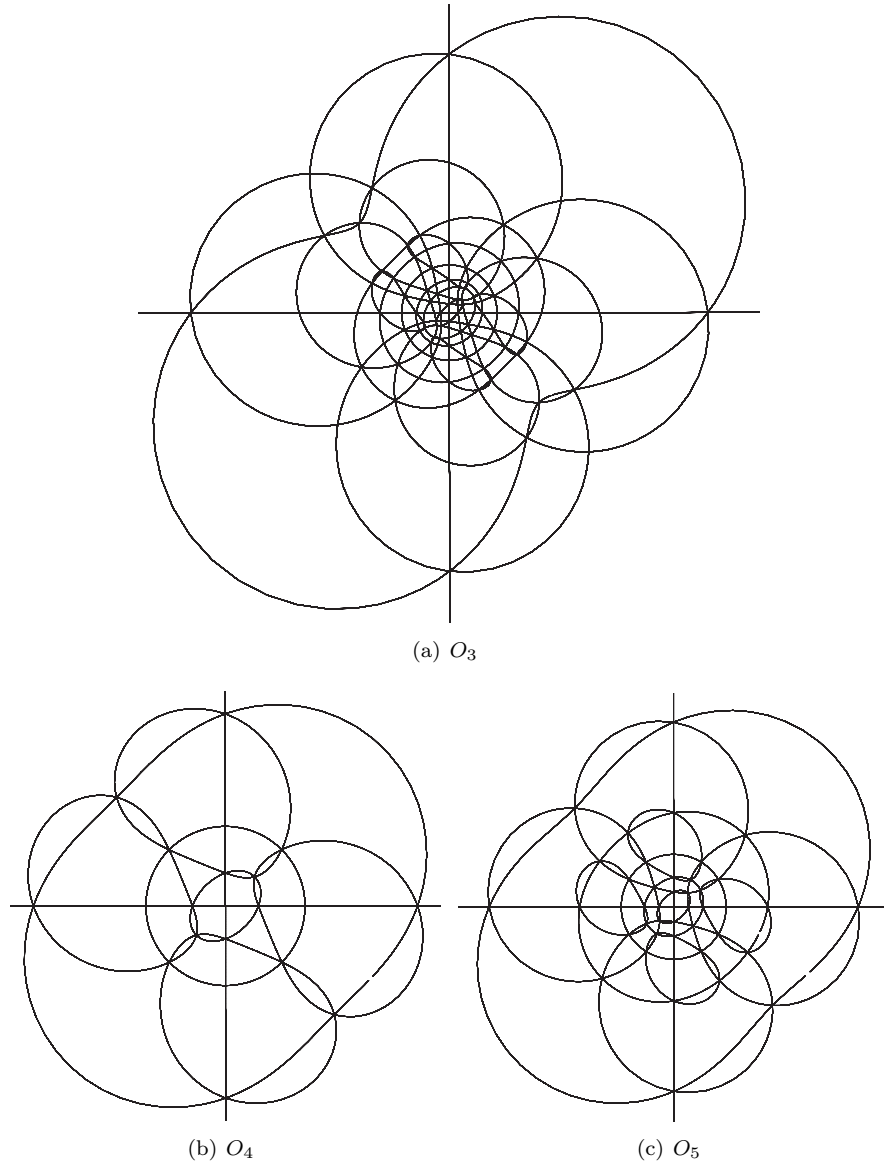


FIGURE 2.

of size 8. The matrix group $\langle A_1, A_2 \rangle$ associated to any point of this orbit is the binary icosahedral group BI_{120} .

40 There are two $\text{Aut}(F_2)$ orbits O_1, O_5 of size 40. These are on levels $(3 + \zeta_5^2 + \zeta_5^3)/4 = 0.34549\dots$ and $(2 - \zeta_5^2 - \zeta_5^3)/4 = 0.904508\dots$ (respectively), where $\zeta_5 = \exp 2\pi i/5$. The action of $\text{Aut}(F_2)$ on either of these orbits of size 40 gives an imprimitive group with ten blocks of size 4, the action on the ten blocks being that of S_{10} . (Points in a block of size 4 are obtained from each other by changing the sign of two coordinates.) The elements of the two sets O_1, O_5 are defined over the cyclotomic field $\mathbb{Q}(\zeta_5)$ and are interchanged by the Galois automorphism $\alpha : \mathbb{Q}(\zeta_5) \rightarrow \mathbb{Q}(\zeta_5), \alpha(\zeta_5) = \zeta_5^2$. The matrix group $\langle A_1, A_2 \rangle$ associated to any point of these orbits is also BI_{120} .

36 There is a single $\text{Aut}(F_2)$ orbit O_2 of size 36. This orbit is on level $3/4$. This case has permutation group of order $2^{16} \cdot 3^4$ and is imprimitive with blocks of sizes 2, 4, 12. The associated group $\langle A_1, A_2 \rangle$ is BO_{48} .

16 There is a single $\text{Aut}(F_2)$ orbit O_4 of size 16. The associated matrix group $\langle A_1, A_2 \rangle$ is BT_{24} and is isomorphic to $\text{SL}(2, \mathbb{F}_3)$. The permutation group has order $2^{12} \cdot 3^2$ and has blocks of size 4. This orbit is on level $1/2$.

Any point $p \in \partial\mathcal{T}$ with finite $\text{Aut}(F_2)$ -orbit has the form $\Pi(r/n, s/n)^T$ for some $(r/n, s/n)^T$ in \mathbb{Q}^2 . Then the action of $\text{Aut}(F_2)$ on p is determined by the action of $\text{SL}(2, \mathbb{Z})$ on $(r/n, s/n)^T$, this action being given by $\text{SL}(2, \mathbb{Z}_n)$. Thus the kernel of the permutation action on all such points is a congruence subgroup.

We may also discuss the $\text{Aut}(F_2)$ -action on the finite orbits O_1, O_2, O_3, O_4, O_5 in the following way.

If $|y| \leq 1$ then the restriction of the map $\sigma_1 : (x, y, z) \mapsto (z, y, 2yz - x)$ to the affine plane $y = \cos(2\pi\rho_y)$ is topologically conjugate to a rotation through the angle $2\pi\rho_y$, because the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 2y \end{pmatrix}, \begin{pmatrix} \cos(2\pi\rho_y) & -\sin(2\pi\rho_y) \\ \sin(2\pi\rho_y) & \cos(2\pi\rho_y) \end{pmatrix}$$

are similar. Therefore the ellipse $x^2 + z^2 - 2xyz = t - y^2$, in which the plane $y = \cos(2\pi\rho_y)$ meets E_t , is mapped to itself by σ_1 with rotation number ρ_y . Thus points of the finite orbits O_1, O_2, O_3, O_4, O_5 must have y (and similarly x, z) of the form $\cos(2\pi p/q)$. The following table lists the number of points of each O_i in each such ellipse (including the case $\rho_0 = \frac{1}{4}$, which is a circle in the coordinate plane $y = 0$). It is these points that are represented in Figures 1 and 2 as the triple points of intersection of ellipses (after stereographic projection). Our σ_1 rotates parallel slices by related amounts, in contrast to the Rubik cube in which these slices can be rotated independently! We indicate this in the following table.

ρ_y	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	Total	t	Typical points
O_1			6	10	8	10	6			40	$\frac{5+\sqrt{5}}{8}$	$(\frac{1}{2}, \frac{\sqrt{5}+1}{4}, \frac{\sqrt{5}+1}{4})^T$ $(\frac{\sqrt{5}+1}{4}, \frac{\sqrt{5}+1}{4}, \frac{\sqrt{5}+1}{4})^T$
O_2		8	6		8		6	8		36	$\frac{3}{4}$	$(\frac{1}{2}\sqrt{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})^T$ $(0, \frac{1}{2}, \frac{1}{2}\sqrt{2})^T$
O_3	10		12	10	8	10	12		10	72	$\frac{3}{4}$	$(\frac{1}{2}, \frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}+1}{4})^T$ $(0, \frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}+1}{4})^T$
O_4			6		4		6			16	$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ $(\frac{1}{2}, 0, \frac{1}{2})^T$
O_5			6	10	8	10	6			40	$\frac{5-\sqrt{5}}{8}$	$(\frac{1}{2}, \frac{1-\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4})^T$ $(\frac{1-\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4})^T$

2. PRELIMINARIES FOR $n = 3$

As we have seen in the above, in the $n = 2$ case there is an invariant of the action $E(x, y, z) = x^2 + y^2 + z^2 - 2xyz$ that comes from $\text{trace}(a_1^{-1}a_2^{-1}a_1a_2)$. This ‘Fricke character’ however is also determined by the fact that for real θ_1, θ_2 the values $x = \cos \theta_1, y = \cos \theta_2, z = \cos(\theta_1 + \theta_2)$ satisfy the relation $E(x, y, z) = 1$. In fact the relation $E(x, y, z) - 1 = 0$ generates the ideal of polynomials $p \in \mathbb{Q}[x, y, z]$ such that $p(\cos \theta_1, \cos \theta_2, \cos(\theta_1 + \theta_2)) = 0$ for all $\theta_1, \theta_2 \in \mathbb{R}$.

Further, the relevance of $x = \cos \theta_1, y = \cos \theta_2, z = \cos(\theta_1 + \theta_2)$ is that they are the half-traces of the matrices $A_i = \begin{pmatrix} \cos \theta_i + i \sin \theta_i & 0 \\ 0 & \cos \theta_i - i \sin \theta_i \end{pmatrix}, i = 1, 2$ and $A_1 A_2 = \begin{pmatrix} \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) & 0 \\ 0 & \cos(\theta_1 + \theta_2) - i \sin(\theta_1 + \theta_2) \end{pmatrix}$. We note that the triple $A_1, A_2, A_1 A_2$ pair-wise commute. Thus they do not give generic values for x, y, z . However they give special values of x, y, z that satisfy $E = 1$.

In the case $n = 3$ that we are considering next these two different ways of understanding E give two different algebraic objects, one of which will be an invariant for the action of $\text{Aut}(F_3)$, while the other will turn out to be an ideal.

To find the ideal just referred to we consider three diagonal commuting matrices: If $A_i = \begin{pmatrix} \cos \theta_i + i \sin \theta_i & 0 \\ 0 & \cos \theta_i - i \sin \theta_i \end{pmatrix}$, for $i = 1, 2, 3$, then $x_i = \cos \theta_i, x_{ij} = \cos(\theta_i + \theta_j), x_{ijk} = \cos(\theta_i + \theta_j + \theta_k)$. We wish to find the ideal of relations satisfied by these x_i, x_{ij}, x_{ijk} . Then in this situation $x_i = \cos \theta_i, i = 1, 2, 3$, and we let $y_i = \sin \theta_i, i = 1, 2, 3, x_{ij} = \cos(\theta_i + \theta_j), i, j = 1, 2, 3$, and similarly $y_{ij} = \sin(\theta_i + \theta_j), i, j = 1, 2, 3, x_{ijk} = \cos(\theta_i + \theta_j + \theta_k), i = 1, 2, 3$, and $y_{ijk} = \sin(\theta_i + \theta_j + \theta_k), i, j, k = 1, 2, 3$.

Let \mathcal{I} be the ideal of $\mathbb{Q}[x_1, x_2, \dots, x_{123}, y_1, y_2, \dots, y_{123}]$ consisting of all polynomials $f(x_1, x_2, \dots, x_{123}, y_1, y_2, \dots, y_{123})$ such that

$$f(\cos \theta_1, \cos \theta_2, \dots, \cos(\theta_1 + \theta_2 + \theta_3), \sin \theta_1, \sin \theta_2, \dots, \sin(\theta_1 + \theta_2 + \theta_3)) = 0.$$

Then using basic trigonometric identities we see that \mathcal{I} contains the polynomials

$$\begin{aligned} x_1^2 + y_1^2 - 1, \quad x_2^2 + y_2^2 - 1, \\ x_3^2 + y_3^2 - 1, \quad x_{12}^2 + y_{12}^2 - 1, \\ x_{13}^2 + y_{13}^2 - 1, \quad x_{23}^2 + y_{23}^2 - 1, \quad x_{123}^2 + y_{123}^2 - 1, \\ x_{12} - x_1x_2 + y_1y_2, \quad x_{23} - x_2x_3 + y_2y_3, \\ x_{13} - x_1x_3 + y_1y_3, \quad y_{12} - y_1x_2 - x_1y_2, \\ y_{23} - y_2x_3 - x_2y_3, \quad y_{13} - y_1x_3 - x_1y_3, \\ x_{123} - x_1x_{23} + y_1y_{23}, \quad y_{123} - y_1x_{23} - x_1y_{23}. \end{aligned}$$

Finding a Gröbner basis for \mathcal{I} (using [2]), and then doing an elimination we obtain the following relations among the x_i, x_{ij}, x_{ijk} :

$$\begin{aligned} x_3^2 - 2x_3x_{12}x_{123} + x_{12}^2 + x_{123}^2 - 1, \\ x_2^2 - 2x_2x_{13}x_{123} + x_{13}^2 + x_{123}^2 - 1, \\ x_2x_3x_{123}^2 - \frac{1}{2}x_2x_3 - \frac{1}{2}x_2x_{12}x_{123} - \frac{1}{2}x_3x_{13}x_{123} + \frac{1}{2}x_{12}x_{13} - \frac{1}{2}x_{23}x_{123}^2 + \frac{1}{2}x_{23}, \\ x_1 + 2x_2x_3x_{123} - x_2x_{12} - x_3x_{13} - x_{23}x_{123}, \\ x_2x_{12}x_{23} - 2x_2x_{13}x_{123}^2 + x_2x_{13} + 2x_3x_{12}x_{123}^2 - x_3x_{12} - x_3x_{13}x_{23} - x_{12}^2x_{123} + x_{13}^2x_{123}, \\ x_2x_3x_{12} - 2x_2x_{123}^3 + x_2x_{123} - x_3x_{23}x_{123} - x_{12}^2x_{13} + 2x_{12}x_{23}x_{123}^2 - x_{12}x_{23} + x_{13}x_{123}^2, \\ x_2x_3x_{23} - x_2x_{13}x_{123} - x_3x_{12}x_{123} + \frac{1}{2}x_{12}^2 + \frac{1}{2}x_{13}^2 - \frac{1}{2}x_{23}^2 + x_{123}^2 - \frac{1}{2}. \end{aligned}$$

Let $R = \mathbb{Q}[x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]$ and let $\mathcal{X} \subset R$ denote the ideal generated by these polynomials. One finds that \mathcal{X} has dimension 3 with x_{13}, x_{23}, x_{123} being algebraically independent. The ideal \mathcal{X} is one of the analogues of the Fricke character E from the $n = 2$ situation.

We define the following automorphisms of F_3 :

$$\begin{aligned} U : [a_1, a_2, a_3] &\mapsto [a_1a_2, a_2, a_3]; \\ Q : [a_1, a_2, a_3] &\mapsto [a_2, a_3, a_1]; \\ S : [a_1, a_2, a_3] &\mapsto [a_1^{-1}, a_2, a_3] \\ P : [a_1, a_2, a_3] &\mapsto [a_2, a_1, a_3]. \end{aligned}$$

It is well-known [21, p. 164] that $\text{Aut}(F_3)$ is generated by U, P, S, Q . The action (on the left) of $\text{Aut}(F_3)$ on R is determined by the actions of the generators U, P, S, Q :

$$\begin{aligned} U([x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]) &= [x_{12}, x_2, x_3, 2x_2x_{12} - x_1, x_{123}, x_{23}, 2x_2x_{123} - x_{13}]; \\ Q([x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]) &= [x_2, x_3, x_1, x_{23}, x_{12}, x_{13}, x_{123}]; \\ S([x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]) &= [x_1, x_2, x_3, 2x_1x_2 - x_{12}, 2x_1x_3 - x_{13}, x_{23}, 2x_1x_{23} - x_{123}]; \\ P([x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]) &= [x_2, x_1, x_3, x_{12}, x_{23}, x_{13}, -4x_1x_2x_3 + 2x_1x_{23} + 2x_2x_{13} + 2x_3x_{12} - x_{123}]. \end{aligned}$$

This also gives the actions of U, P, S, Q (on the right) on elements of \mathbb{R}^7 . We now show that this really does determine an action of $\text{Aut}(F_3)$ on R and on \mathbb{R}^7 :

Lemma 2.1. *The above action of the generators U, P, S, Q determines a homomorphism $\text{Aut}(F_3) \rightarrow \text{Aut}(R)$ and an anti-homomorphism $\text{Aut}(F_3) \rightarrow \text{Homeo}(\mathbb{R}^7)$, such that for all $\sigma \in \text{Aut}(F_3)$ we have*

$$\sigma(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\sigma.$$

Further, the ideal \mathcal{X} is invariant under this action.

Proof A presentation for $\text{Aut}(F_3)$ is given in [21, p. 164]:

$$\begin{aligned} \langle U, Q, P, S | P^2, Q^3, S^2, (QP)^2, SQP = QPS, SQ^{-1}PQ = Q^{-1}PQS, \\ SQ^{-1}SQ = Q^{-1}SQS, (U, QPQ^{-1}PQ), (U, Q^{-2}SQ^2), (PSPU)^2, \\ PUPSUSPS = U, (PQ^{-1}UQ)^2UQ^{-1} = UQ^{-1}U, USUS = SUSU, \\ (U, PQ^{-1}SUSQP), (U, PQ^{-1}PQPUPQ^{-1}PQP) \rangle. \end{aligned}$$

(The presentation from [21, p. 164] seems to have two further relations, however these relations only apply to the situation $n > 3$.) Thus to check that we do obtain a homomorphism $\text{Aut}(F_3) \rightarrow \text{Aut}(R)$ we just need to show that each relation in the presentation acts trivially on R . This is straight-forward. The fact that $\sigma(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\sigma$ is proved by induction on the length of σ as a word in the generators U, Q, S, P .

Now to check that the ideal \mathcal{X} is invariant under this action we just show that for each element x of a basis for \mathcal{X} we have $U(x), P(x), S(x), Q(x) \in \mathcal{X}$. This is also straight-forward. \square

We will need the following result using an element discovered by Horowitz [13]; this gives us an element that is fixed by $\text{Aut}(F_3)$ (thus this element also corresponds to E in the $n = 2$ case):

Lemma 2.2. *The element*

$$\begin{aligned} F = x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 + x_{123}^2 + 4x_1x_2x_3x_{123} \\ - 2x_1x_2x_{12} - 2x_1x_3x_{13} - 2x_1x_{23}x_{123} - 2x_2x_3x_{23} - 2x_2x_{13}x_{123} - 2x_3x_{12}x_{123} \\ + 2x_{12}x_{13}x_{23} - 1, \end{aligned}$$

is in \mathcal{X} and is fixed by each $\alpha \in \text{Aut}(F_3)$.

Proof For the first statement we use a Gröbner basis for \mathcal{X} to show that $F \in \mathcal{X}$. For the last statement we just show that $U(F) = P(F) = S(F) = Q(F) = F$. \square

Let $V(\mathcal{X})$ denote the (real) variety corresponding to \mathcal{X} , and let $V(F)$ be that for F .

We note that by the results of [13], the polynomial F generates the principal ideal of all trace relations for three matrices in $\text{SL}(2, \mathbb{C})$. Thus for all trace septuples $p = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ determined by a triple $A_1, A_2, A_3 \in \text{SL}(2, \mathbb{C})$ we have $F(p) = 0$ i.e. $p \in V(F)$. Thus in what follows we will always assume that $p \in V(F)$.

We have already noted that \mathcal{X} has dimension 3. This next result gives a parametrization of a compact part of $V(\mathcal{X})$ of dimension 3 that corresponds to the curvilinear tetrahedron \mathcal{T} in the $n = 2$ case. For $n \in \mathbb{N}$ we let \mathbb{T}^n denote the n -torus $(S^1)^n$.

Lemma 2.3. Define $\Pi_3 : \mathbb{R}^3 \rightarrow V(\mathcal{X})$, by

$$(t_1, t_2, t_3) \mapsto (\cos 2\pi t_1, \cos 2\pi t_2, \cos 2\pi t_3, \cos 2\pi(t_1 + t_2), \cos 2\pi(t_1 + t_3), \\ \cos 2\pi(t_2 + t_3), \cos 2\pi(t_1 + t_2 + t_3)).$$

Then Π_3 factors through

$$\mathbb{R}^3 \rightarrow \mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3 \rightarrow \mathbb{T}^3/(\pm 1) \rightarrow V(\mathcal{X}),$$

and has image homeomorphic to $\mathbb{T}^3/(\pm 1)$.

Proof To check that the image of Π_3 is in $V(\mathcal{X})$ we just take each element $x = x(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ of a basis for \mathcal{X} and show that $x(\Pi_3(t_1, t_2, t_3)) = 0$. This is straightforward. The rest follows. \square

Next we determine some copies of \mathbb{R}^3 inside \mathbb{R}^7 . If we choose matrices

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we let $y_i = \text{trace}(A_i)/2$, $y_{ij} = \text{trace}(A_i A_j)/2$, $y_{ijk} = \text{trace}(A_i A_j A_k)/2$, then

$$y_3 = y_{13} = y_{23} = y_{123} = 0.$$

We say that $(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123})$ has zeros in positions 3, 5, 6, 7. We have:

Lemma 2.4. If $\alpha \in \text{Aut}(F_3)$ and $y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}$ are as defined above, then $(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123})\alpha$ has zeros in one of the sets

$$\{1, 2, 3, 7\}, \{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{1, 4, 5, 7\}, \{2, 3, 4, 5\}, \{2, 4, 6, 7\}, \{3, 5, 6, 7\}.$$

This gives a homomorphism $\Sigma_7 : \text{Aut}(F_3) \rightarrow S_7$, where

$$\begin{aligned} \Sigma_7(U) &= (1, 5)(2, 6); & \Sigma_7(Q) &= (2, 3, 5)(4, 7, 6); \\ \Sigma_7(S) &= 1; & \Sigma_7(P) &= (3, 5)(4, 6). \end{aligned}$$

Lastly, $\langle \Sigma_7(U), \Sigma_7(Q), \Sigma_7(S), \Sigma_7(P) \rangle \cong \text{SL}(3, 2)$.

Proof One checks that elements $(y_1, y_2, \dots, y_{123}) \in \mathbb{R}^7$ having zero in one of these sets are permuted by $\text{Aut}(F_3)$. Thus we obtain the (transitive) permutation representation Σ_7 . The rest is a calculation. \square

Let $K_7 = \ker(\Sigma_7)$. In [21] we can find a presentation for $\text{Aut}(F_3)$, and since K_7 has index 168 we can use the Reidemeister-Schreier process (as implemented in [2], for example), to find generators for K_7 . These are

$$(2.1) \quad \begin{aligned} &S, USU^{-1}, U^2, PUSU^{-1}P, PU^2P, QUSU^{-1}Q^{-1}, (Q^{-1}UQ)^2, PQUSU^{-1}Q^{-1}P, \\ &PQU^2Q^{-1}P, PQ^{-1}U^2QP, (Q^{-1}UQ^{-1}UQU^{-1}Q)^2, (UQ^{-1}UQUQ^{-1}U^{-1}QU^{-1})^2, \\ &(UQ^{-1}UQ^{-1}UQU^{-1}QU^{-1})^2, Q^{-1}UPQUQU^{-1}Q^{-1}U^{-1}QU^{-1}P, \\ &PQUQ^{-1}UPUQU^{-1}Q^{-1}PU^{-1}, PUPQUUPU^{-1}QU^{-1}P, \\ &PQ^{-1}UQ^{-1}UPUQU^{-1}Q^{-1}PU^{-1}Q, (QUQ^{-1}UQ^{-1}UQU^{-1}QU^{-1}Q)^2. \end{aligned}$$

There are 18 generators in this list of generators and $K_7/K'_7 \cong \mathcal{C}_2^{18}$, so that we cannot generate K_7 by less than 18 elements.

For generic $x, y, z \in \mathbb{R}$ define

$$\begin{aligned} u_1 &= [x, y, 0, z, 0, 0, 0]; & u_2 &= [x, 0, y, 0, z, 0, 0]; \\ u_3 &= [0, x, y, 0, 0, z, 0]; & u_4 &= [x, 0, 0, 0, 0, y, z]; \\ u_5 &= [0, x, 0, 0, y, 0, z]; & u_6 &= [0, 0, x, y, 0, 0, z]; \\ u_7 &= [0, 0, 0, x, y, z, 0]. \end{aligned}$$

Let $\mathcal{U}_i \subset \mathbb{R}^7$, $1 \leq i \leq 7$, denote the 3-dimensional subset determined by u_i , so that for example $\mathcal{U}_1 = \{(x, y, 0, z, 0, 0, 0) \in \mathbb{R}^7, x, y, z \in \mathbb{R}\}$. From Lemma 2.4 we see that the action of $\text{Aut}(F_3)$ permutes the \mathcal{U}_i .

One also sees that $F(u_i)$, $1 \leq i \leq 6$, is equal to $E(x, y, z) - 1$, while $F(u_7) = E(x, y, -z) - 1$. This shows that each $V(F) \cap \mathcal{U}_i = \partial \mathcal{T}_i$, $1 \leq i \leq 7$, is a copy of $\partial \mathcal{T}$. From Lemma 2.4, and the fact that $\mathcal{U}_i \cap V(F) = \partial \mathcal{T}_i$, we see that the action of $\text{Aut}(F_3)$ permutes the \mathcal{T}_i with the action given by Lemma 2.4.

Let $\Gamma_{\mathcal{U}}$ be the graph whose vertices are $\mathcal{U}_1, \dots, \mathcal{U}_7$, and where we have an edge $\mathcal{U}_i, \mathcal{U}_j$ exactly when $\dim(\mathcal{U}_i \cap \mathcal{U}_j) = 1$.

Lemma 2.5. *The graph $\Gamma_{\mathcal{U}}$ is the Fano plane with seven points and seven lines as depicted in Figure 3 which is drawn on S^2 (each line is represented by a triangle). The group $\text{Aut}(F_3)$ acts transitively on $\mathcal{U}_1, \dots, \mathcal{U}_7$, giving an epimorphism $\text{Aut}(F_3) \rightarrow \text{SL}(3, 2)$.*

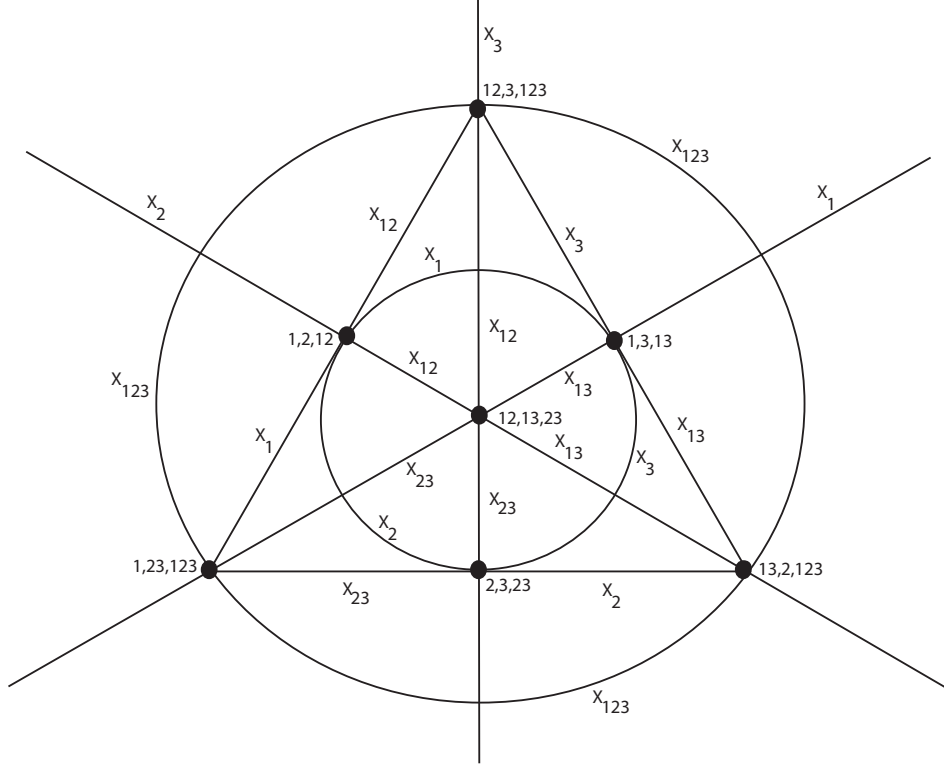


FIGURE 3. The Fano plane.

Proof In Figure 3 we have drawn the graph $\Gamma_{\mathcal{U}}$, indicating each vertex \mathcal{U}_i by the corresponding non-zero coordinates. For example, vertices labeled 12, 3, 123 (corresponding to \mathcal{U}_6) and 1, 2, 12 (corresponding to \mathcal{U}_1) are drawn with an edge between them since they share the coordinate 12, and $\mathcal{U}_6 \cap \mathcal{U}_1$ is the x_{12} -axis. The rest is clear. \square

We have now developed all the ideas necessary for us to be able to state our main result in the case $n = 3$:

Theorem 2. *If $p \in \mathcal{P}_3 \cap V(F)$, then we have one of the following:*

- (i) p is on a coordinate axis;
 - (ii) there is some $1 \leq i \leq 7$ such that $p \in \partial\mathcal{T}_i$;
 - (iii) $p \in \text{Image}(\Pi_3)$;
 - (iv) the associated group $\langle A_1, A_2, A_3 \rangle$ is finite.
- In particular, $\mathcal{F}_3 = \mathcal{P}_3$.

We will say more about (ii) and (iii) below.

3. PERIODIC POINTS ON E_1 AND ASSOCIATED MATRIX GROUPS FOR POINTS NOT ON E_1

We first characterise those points $p \in \mathcal{P}_2$ which are on E_1 , noting that the set E_1 contains $\partial\mathcal{T}$.

Lemma 3.1. (i) *A point $p \in \partial\mathcal{T}$ is in \mathcal{P}_2 if and only if there is $(\theta_1, \theta_2)^T \in \mathbb{Q}^2/\mathbb{Z}^2$ such that $\Pi_2(\theta_1, \theta_2)^T = p$.*

(ii) *Let $p \in E_1$. Then $p \in \mathcal{P}_2$ if and only if $p \in \partial\mathcal{T} \cap \mathcal{P}_2$.*

Proof. (i) First suppose that $\Pi_2(\theta_1, \theta_2)^T = p$, where $(\theta_1, \theta_2)^T \in \mathbb{Q}^2/\mathbb{Z}^2$. This means that we can write $\theta_1 = s/n, \theta_2 = t/n$ where $s, t, n \in \mathbb{Z}, n > 0$. Now given $\sigma \in \text{Aut}(F_2)$ there is $u \in \mathbb{N}$ such that $\Phi_2(\sigma^u) \equiv I_2 \pmod{n}$. This means that the action of $\Phi_2(\sigma^u)$ on \mathbb{T}^2 fixes $(\theta_1, \theta_2)^T$. Since the action of σ on $\partial\mathcal{T}$ is determined by the action of $\Phi_2(\sigma)$ on \mathbb{T}^2 it follows that σ^u fixes $\Pi_2(\theta_1, \theta_2)^T = p$, as required.

For the converse of (i) suppose that $(\theta_1, \theta_2)^T \in \mathbb{T}^2 \setminus (\mathbb{Q}^2/\mathbb{Z}^2)$; in fact we may assume that $\theta_1 \notin \mathbb{Q}$ (the other case being similar). Then from the action of $\Phi_2(\sigma_2)$ and (1.1) above we have $\Phi_2(\sigma_2^k)(\theta_1, \theta_2)^T = (\theta_1, \theta_2 - k\theta_1)^T$ and so $\Phi_2(\sigma_2^k)(\theta_1, \theta_2)^T \not\equiv (\theta_1, \theta_2)^T \pmod{\mathbb{Z}^2}$ for all $k \in \mathbb{N}$; this shows that $(\theta_1, \theta_2)^T$ has an infinite orbit under the action of $\langle \sigma_2 \rangle$ and concludes the proof of (i).

(ii) We now consider those points $p \in E_1 \cap \mathcal{P}_2$ not on $\partial\mathcal{T}$. Any such point is on one of the cones which meet $\partial\mathcal{T}$ at a point of V . Since $\text{Aut}(F_2)$ acts transitively on these cones we may assume that p is on the cone \mathcal{C} which meets $(1, 1, 1)^T \in V$. It is easy to check that this cone can be parameterised by the function

$$\Pi' : \mathbb{R}^2 \rightarrow \mathcal{C}; \quad \Pi'(t_1, t_2)^T = (\cosh t_1, \cosh t_2, \cosh(t_1 + t_2))^T.$$

The function Π' is a 2 to 1 cover branched at $(0, 0)^T$, the point corresponding to $(1, 1, 1)^T \in V$, and satisfies $\Pi'((t_1, t_2)^T) = \Pi'((-t_1, -t_2)^T)$. As in (1.1) the action of $\text{Aut}(F_2)$ on \mathcal{C} is induced from an action of $\text{GL}(2, \mathbb{Z})$ on \mathbb{R}^2 :

$$(\Pi'(t_1, t_2)^T)\alpha = \Pi'(\Phi_2(\alpha)(t_1, t_2)^T).$$

Thus the point $p \in \mathcal{C} \setminus V$ would correspond to $(t_1, t_2)^T \in \mathbb{R}^2 \setminus \{(0, 0)^T\}$ and it is now easy to see that either $\Phi_2(\sigma_1)$ or $\Phi_2(\sigma_2)$ has infinite orbit on $(t_1, t_2)^T$, so p has infinite orbit under either $\langle \sigma_1 \rangle$ or $\langle \sigma_2 \rangle$. This proves (ii). \square

Given $p = (x, y, z)^T \in \mathbb{R}^3 \setminus E_1$, with associated matrices A_1, A_2 , defined up to conjugacy by p , we define the homomorphism

$$\mu = \mu(p) : F_2 \rightarrow \langle A_1, A_2 \rangle, \quad \mu(a_1) = A_1, \quad \mu(a_2) = A_2.$$

Given $\sigma \in \text{Aut}(F_2)$ we obtain matrices $C = \mu(\sigma(a_1)), D = \mu(\sigma(a_2))$. We will denote $\mu(\sigma(a_1))$ by $\sigma(A_1)$ and $\mu(\sigma(a_2))$ by $\sigma(A_2)$. Then $\langle A_1, A_2 \rangle = \langle \sigma(A_1), \sigma(A_2) \rangle$.

The following shows that if $p \in \mathcal{P}_2$, then A_1, A_2 have finite order under certain simple conditions.

Lemma 3.2. *Let $p \in \mathcal{P}_2, p \notin E_1$, and also assume that p does not lie on a coordinate axis. Let $A_1 = A_1(p), A_2 = A_2(p)$ be the associated matrices. Then A_2 has finite order, and for any $\sigma \in \text{Aut}(F_2)$ the matrix $\sigma(A_2)$ has finite order.*

Proof. The result for arbitrary σ will follow from the case $\sigma = 1$. :et $p = (x, y, z)^T$.

First note that $\text{trace}(A_2) = 2y$ where $p = (x, y, z)^T$. Now, since $M \in \text{SL}(2, \mathbb{C})$ has finite order if and only if $\text{trace}(M) = 2 \cos 2\pi q$, for some $q \in \mathbb{Q}$ (except when this value is ± 2 , when we might have a parabolic, up to sign), we will next show that $2y$ is of this form.

From (0.1) we have: if $n \in \mathbb{Z}$, then

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix}^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1^n;$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x & -1 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_2^n.$$

Let $M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix}$, so that $M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1$, and let $\lambda, 1/\lambda$ be the eigenvalues of the submatrix $M' = \begin{pmatrix} 0 & 1 \\ -1 & 2y \end{pmatrix}$. Now $\text{trace}(M') = 2y$, where $y = \text{trace}(A_2)/2$. Since $p \in \mathcal{P}_2$ we see that there is $n \in \mathbb{N}$ such that $(p)\sigma_1^n = p$ and so $(M')^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now if $x = z = 0$, then $p = (x, y, z)^T$ is on a coordinate axis. Thus we may assume that $(x, z)^T \neq (0, 0)^T$. It follows now that 1 is an eigenvalue of $(M')^n$, showing that $\lambda^n = 1$. Thus λ is a root of unity and $2y = \lambda + 1/\lambda = 2 \cos 2\pi q$, for some $q \in \mathbb{Q}$. We distinguish two cases:

Case 1: $\lambda \neq \pm 1$ Here the fact that $\text{trace}(A_2) = \text{trace}(M') = 2y = \lambda + 1/\lambda = 2 \cos 2\pi q, q \in \mathbb{Q}$, where $\lambda \neq 1/\lambda$, shows that M' is diagonalizable and of finite order, so that A_2 has finite order.

Case 2: $\lambda = \pm 1$ Here we may assume that $A_2 = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, a = \pm 1$. There are two sub-cases: (i) $b \neq 0$; and (ii) $b = 0$.

If we have (i) $b \neq 0$, then we let $A_1 = \begin{pmatrix} u & v \\ r & s \end{pmatrix}$, so that $\text{trace}(A_1 A_2^m) = a^m(s + u + vabm)$, which takes on infinitely many values for $m \in \mathbb{N}$ if $v \neq 0$, contradicting $p \in \mathcal{P}_2$. Thus $v = 0$, from which it follows that A_1 and A_2 have a common eigenvector and so $p \in E_1$, a contradiction.

If we have (ii), then from $\text{trace}(A_1) = 2x$ we get $\text{trace}(A_1 A_2) = 2ax$ and $p = (x, a, ax) \in E_1$ since $a = \pm 1$. This gives Lemma 3.2. \square

Corollary 3.3. *Let $p \in \mathcal{P}_2$ and assume that p is not on level 1 and not on a coordinate axis. Let A_1, A_2 be the associated matrices. Then each of*

$$A_1 A_2^h, A_2 A_1^h, A_1 (A_1 A_2)^h, A_2 (A_1 A_2)^h, h \in \mathbb{Z},$$

has finite order.

Proof. It is easy to see that each of these elements has the form $\sigma(A_2)$ for some $\sigma \in \text{Aut}(F_2)$, and so the result follows from Lemma 3.2. \square

Recall that the points of V are permuted transitively by $\text{Aut}(F_2)$. We will need:

Lemma 3.4. *Let $p \in E_1 \cap \mathcal{P}_2 \setminus V$ correspond to the matrices A_1, A_2 . Then*

- (i) A_1, A_2 are conjugate to a pair of lower-triangular matrices.
- (ii) *There is some element in the $\text{Aut}(F_2)$ -orbit of p that has 1 as an entry.*

Proof Let $p = (x_1, x_2, x_{12})$. (i) First assume that one of $A_1, A_2, A_1 A_2$ has distinct eigenvalues. Without loss of generality we assume that A_1 has eigenvalues $\lambda \neq 1/\lambda$. Then, by a conjugacy, we can further assume $A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We calculate $x_1 = \text{trace}(A_1)/2, x_2 = \text{trace}(A_2)/2, x_{12} = \text{trace}(A_1 A_2)/2$, and find that

$$E(x_1, x_2, x_{12}) - 1 = \frac{1}{4\lambda^2} (1 - \lambda^2)^2 (1 - ad).$$

Since we have $E(x_1, x_2, x_{12}) - 1 = 0$ we thus have either (i) $\lambda = \pm 1$; or (ii) $ad = 1$. If we have (ii), then $\det(A_2) = 1$ gives $bc = 0$, and we are done. If we have (i), then $\lambda = 1/\lambda$, a contradiction.

Thus we may now assume that each of $A_1, A_2, A_1 A_2$ do not have distinct eigenvalues, so that $x_1, x_2, x_{12} \in \{\pm 1\}$. But in this case we must have $p \in V$, a contradiction.

(ii) From (i) we can take $A_i = \begin{pmatrix} \lambda_i & 0 \\ c_i & 1/\lambda_i \end{pmatrix}, i = 1, 2$. We claim that the λ_i are roots of unity. Since $p \in \mathcal{P}_2$ there $0 < r_1 < r_2 < r_3$ such that $\text{trace}(A_1^{r_1} A_2) = \text{trace}(A_1^{r_2} A_2) = \text{trace}(A_1^{r_3} A_2)$. Let $1 \leq i \neq j \leq 3$. Solving $\text{trace}(A_1^{r_i} A_2) = \text{trace}(A_1^{r_j} A_2)$ gives

$$(\lambda_1^{r_i} - \lambda_1^{r_j})(\lambda_2^2 \lambda_1^{r_i + r_j} - 1) = 0.$$

If $\lambda_1^{r_i} - \lambda_1^{r_j} = 0$, then λ_1 is a root of unity. If not, then we must have $\lambda_1^{r_i + r_j} = \lambda_2^{-2}$ for all such $i, j = 1, 2, 3$. But then $\lambda_1^{r_1 + r_2} = \lambda_1^{r_1 + r_3} = \lambda_2^{-2}$, so that λ_1 is again a root of unity. This proves the claim.

Since the multiplicative group $\langle \lambda_1, \lambda_2 \rangle \leq \mathbb{C}^\times$ is a finite subgroup of a field we see that it is cyclic, and so is generated by λ_1, λ_2 or $\lambda_1 \lambda_2$. If it is generated by λ_1 , then there is some $u \in \mathbb{N}$ such that $\text{trace}(A_2 A_1^u) = 2$, and we have found an element of the orbit of p that has 1 as an entry. If $\langle \lambda_1, \lambda_2 \rangle$ is generated by λ_2 or $\lambda_1 \lambda_2$, then we can similarly obtain condition (ii). \square

We will have need of the following result [8, Algebraic Lemma, p. 101]. An *admissible triple* is a triple $(a, b, c) \in \mathbb{R}^3$ where at most one of a, b, c is zero.

Lemma 3.5. *Let $(a, b, c) \in \mathbb{R}^3$ be an admissible triple satisfying*

$$a^2 + b^2 + c^2 - 2abc > 1, \text{ and } |a|, |b|, |c| \leq 1.$$

Then there is $\beta \in B_3$ such that the absolute value of one of the coordinates of $(a, b, c)\beta$ is greater than 1. \square

Proof of Theorem 1

Let $p = (x, y, z) \in \mathcal{P}_2$ where $p \notin V$ is not on a coordinate axis.

If $p \in E_1$, then by Lemma 3.4 we can assume that (up to a conjugacy) we have $A_1 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, A_2 = \begin{pmatrix} \lambda & 0 \\ d & 1/\lambda \end{pmatrix}$. Since the set of traces of the elements $A_1 A_2^n, n \in \mathbb{Z}$, is finite, we see that there are infinitely many $i, j \in \mathbb{N}, i \neq j$, such that $\lambda^i + 1/\lambda^i = \lambda^j + 1/\lambda^j$; but then

$$0 = \lambda^i + 1/\lambda^i - \lambda^j - 1/\lambda^j = (\lambda^{i+j} - 1)(\lambda^i - \lambda^j)/(\lambda^{i+j}),$$

which shows that λ is a root of unity. Thus we have case (3) of Theorem 1.

Now assume that $p = (x, y, z) \in E_t, t > 1$. Then by Lemma 3.5 we may assume that one of x, y, z has absolute value greater than 1. In fact we may assume that $|y| > 1$. But this contradicts Lemma 3.2 which shows that $|y| = |\cos(\theta)|$ for some $\theta \in \mathbb{R}$.

We are left with the cases $p \in E_t, t < 1$, which are dealt with in [8]. \square

4. PERIODIC POINTS IN $\text{Image}(\Pi_3)$ AND $V(\mathcal{X})$

As in the $n = 2$ case we note that each element of $\Pi_3((\mathbb{Q}/\mathbb{Z})^3)$ is in \mathcal{F}_3 .

The analogue of Lemma 3.1 for the situation where $n = 3$ is

Lemma 4.1. (i) *A point $p \in \text{Image}(\Pi_3)$ is in \mathcal{P}_3 if and only if there is $(\theta_1, \theta_2, \theta_3)^T \in \mathbb{Q}^3/\mathbb{Z}^3$ such that $\Pi_3(\theta_1, \theta_2, \theta_3)^T = p$.*

(ii) *Let $p \in V(\mathcal{X}) \cap [-1, 1]^7$. Then $p \in \mathcal{P}_3$ if and only if $p \in \text{Image}(\Pi_3) \cap \mathcal{P}_3$.*

Proof The proof of (i) is similar to the proof of Lemma 3.1 (i).

(ii) So let $p = (x_1, x_2, \dots, x_{123})^T \in V(\mathcal{X}) \cap [-1, 1]^7$. Then the fact that $|x_i| \leq 1, i = 1, 2, 3$, means that we can write $x_i = \cos(\theta_i), i = 1, 2, 3$. Now $x_1^2 + x_2^2 + x_{12}^2 - 2x_1x_2x_{12} - 1 \in \mathcal{X}$. Solving this equation for x_{12} we get

$$x_{12} = \cos(\theta_1)\cos(\theta_2) \pm \sin(\theta_1)\sin(\theta_2) = \cos(\theta_1 \mp \theta_2).$$

Thus we can change the sign of one of θ_1, θ_2 (if necessary) so that we have

$$(4.1) \quad x_1 = \cos(\theta_1), \quad x_2 = \cos(\theta_2), \quad x_{12} = \cos(\theta_1 + \theta_2).$$

We similarly have $x_1^2 + x_3^2 + x_{13}^2 - 2x_1x_3x_{13} - 1 \in \mathcal{X}$, which gives $x_{13} = \cos(\theta_1 \mp \theta_3)$. By changing the sign of θ_3 if necessary, we may thus assume that

$$(4.2) \quad x_{13} = \cos(\theta_1 + \theta_3).$$

Now we also have $x_2^2 + x_3^2 + x_{23}^2 - 2x_2x_3x_{23} - 1 \in \mathcal{X}$ which gives $x_{23} = \cos(\theta_2 \pm \theta_3)$. Of course, what we want is $x_{23} = \cos(\theta_2 + \theta_3)$. So assume otherwise. Then we note that

$$2x_1x_2x_3^2 - x_1x_2 - x_1x_3x_{23} - x_2x_3x_{13} - x_3^2x_{12} + x_{12} + x_{13}x_{23} \in \mathcal{X}.$$

Now substituting (4.1), (4.2) and $x_{23} = \cos(\theta_2 - \theta_3)$ into this element gives

$$-\sin(\theta_1)\sin(\theta_2)\sin^2(\theta_3),$$

which is non-zero for generic θ_i , and so we must have

$$x_{23} = \cos(\theta_2 + \theta_3).$$

Now, considering the element $x_1^2 + x_{23}^2 + x_{123}^2 - 2x_1x_{23}x_{123} - 1 \in \mathcal{X}$, we see that $x_{123} = \cos(\pm(\theta_1 \pm (\theta_2 + \theta_3)))$, while considering the element $x_2^2 + x_{13}^2 + x_{123}^2 -$

$2x_2x_{13}x_{123} - 1 \in \mathcal{X}$, we see that $x_{123} = \cos(\pm(\theta_2 \pm (\theta_1 + \theta_3)))$. Thus the only possibility for reconciling these two equations gives $x_{123} = \cos(\pm(\theta_1 + \theta_2 + \theta_3))$. \square

Lemma 4.2. (a) Suppose that we have a triple of matrices $A_1, A_2, A_3 \in \text{SL}(2, \mathbb{C})$ corresponding to the point $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \mathcal{P}_3$, where one of $x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}$ is ± 1 , and the corresponding matrix is a non-identity parabolic (up to sign). Then $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \text{Image}(\Pi_3)$.

(b) If one of $x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}$ is ± 1 , and the corresponding matrix is the identity matrix, then either $\langle A_1, A_2, A_3 \rangle$ is a finite group or $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \partial \mathcal{T}_i$ for some $1 \leq i \leq 7$.

Proof Assume that one of $A_1, A_2, A_3, A_1A_2, A_1A_3, A_2A_3, A_1A_2A_3$ is $\pm K$, where K is the identity or a parabolic. We first show that (after acting by some element of $\text{Aut}(F_3)$) we may assume that $A_1 = \pm K$. If $A_2 = \pm K$ or $A_3 = \pm K$, then using a power of Q we reduce to the case where $A_1 = \pm K$. If $A_1A_2 = \pm K$, then, using the automorphism determined by $[a_1, a_2, a_3] \mapsto [a_1a_2, a_2, a_3]$ will do this case. If we have $A_1A_3 = \pm K$ or $A_2A_3 = \pm K$, then we use a power of Q to reduce to the case $A_1A_2 = \pm K$, and this case follows. If we have $A_1A_2A_3 = \pm K$, then, using an automorphism, we replace (A_1, A_2, A_3) by $(A_1A_2A_3, A_2, A_3)$, and we have done this last case.

Thus we may now suppose that $x_1 = \pm 1$, any other case being similar by the above. Then (up to conjugacy) we have $A_1 = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, $a = \pm 1$. If we have (a),

then $b \neq 0$. Now let $A_2 = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$, so that for all $n \in \mathbb{Z}$ we have $\text{trace}(A_1^n A_2) = a^n(u + x + abnv)$, which forces $v = 0$, since $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \mathcal{P}_3$. Thus A_1 and A_2 are lower-triangular. Now by Corollary 3.3 we see that $\text{trace}(A_2) = 2\cos(2\pi p/q)$, $p/q \in \mathbb{Q}$, and since $\det A_2 = 1$ it follows that $A_2 = \begin{pmatrix} \lambda & 0 \\ c & \bar{\lambda} \end{pmatrix}$, $\lambda = \cos(2\pi p/q) + i\sin(2\pi p/q)$. One similarly shows that A_3 is lower-triangular of this form. It follows that $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \text{Image}(\Pi_3)$. This proves (a).

For (b) we now assume that $A_1 = \varepsilon I_2$, $\varepsilon = \pm 1$. Then

$$(4.3) \quad (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (\varepsilon, x_2, x_3, \varepsilon x_2, \varepsilon x_3, x_{23}, \varepsilon x_{23}).$$

Now the subgroup of $\text{Aut}(F_3)$ that fixes $a_1 \in F_3$ contains a copy of $\text{Aut}(F_2)$ that acts on vectors of the form (4.3) just like $\text{Aut}(F_2)$ acts on the triples $(x_1, x_2, x_{12}) \in \mathbb{R}^3$, so by Theorem 1 we have either

- (i) $\langle A_2, A_3 \rangle$ is a finite group; or
- (ii) (x_2, x_3, x_{23}) is on a coordinate axis of \mathbb{R}^3 ; or
- (iii) $(x_2, x_3, x_{23}) \in \text{Image}(\Pi_2)$.

If we have (i), then $\langle A_1, A_2, A_3 \rangle$ is a finite group, and we are done.

If we have (iii), then we write $x_2 = \cos(\theta_2)$, $x_3 = \cos(\theta_3)$, $x_{23} = \cos(\theta_2 + \theta_3)$, so that if $\varepsilon = 1$, then (4.3) becomes

$$(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (\cos(0), \cos(\theta_2), \cos(\theta_3), \cos(\theta_2), \cos(\theta_3), \cos(\theta_2 + \theta_3), \cos(\theta_2 + \theta_3)) \in \text{Image}(\Pi_3).$$

On the other hand, if $\varepsilon = -1$, then

$$(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (\cos(\pi), \cos(\theta_2), \cos(\theta_3), \cos(\pi + \theta_2), \cos(\pi + \theta_3), \cos(\theta_2 + \theta_3), \cos(\pi + \theta_2 + \theta_3)) \in \text{Image}(\Pi_3).$$

This does case (iii).

If we have (ii), then without loss of generality we may assume that $(x_2, x_3, x_{23}) = (0, 0, x_{23})$. Then from (4.3) we have

$$(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) = (\varepsilon, 0, 0, 0, 0, x_{23}, \varepsilon x_{23}).$$

Then $E(\varepsilon, x_{23}, \varepsilon x_{23}) = 1$ so that $(\varepsilon, 0, 0, 0, 0, x_{23}, \varepsilon x_{23}) \in \partial\mathcal{T}_4$. \square

Lemma 4.3. *Let $p = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) \in \mathcal{P}_3$.*

(i) Suppose that one of $x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}$ has absolute value greater than 1. Then p is on a coordinate axis of \mathbb{R}^7 .

(ii) Suppose that $\alpha \in \text{Aut}(F_3)$ with

$$(p)\alpha = (y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}),$$

where one of $y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}$ has absolute value greater than 1. Then p is on a coordinate axis of \mathbb{R}^7 , and so is $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$.

Proof For (i) we may assume (as in the proof of Lemma 4.2) that $|x_1| > 1$. Then using Theorem 1 applied to the point $q = (x_1, x_2, x_{12}) \in \mathbb{R}^3$, we see that q is on a coordinate axis of \mathbb{R}^3 , which shows that $x_2 = x_{12} = 0$. The same argument applied to the triples (x_1, x_3, x_{13}) and (x_1, x_{23}, x_{123}) yields $x_3 = x_{13} = 0, x_{23} = x_{123} = 0$, which now shows that p is on a coordinate axis of \mathbb{R}^7 . This gives (i).

We note that if $p \in \mathbb{R}^7$ is on a coordinate axis in \mathbb{R}^7 , then so is any $(p)\alpha, \alpha \in \text{Aut}(F_3)$. In fact, $\text{Aut}(F_3)$ acts transitively on the axes, so that (ii) follows from (i). \square

5. BINARY DIHEDRAL GROUPS

In this section we investigate the connection between binary dihedral groups and the points of $\mathcal{P}_3 \cap \mathcal{U}_i \cap V(F) = \mathcal{P}_3 \cap \partial\mathcal{T}_i, 1 \leq i \leq 7$.

A binary dihedral group BD_{2n} of order $4n$ has presentation

$$\langle a, b | a^{2n}, b^4, a^n = b^2, a^b = a^{-1} \rangle.$$

It has a faithful representation in $\text{SL}(2, \mathbb{C})$ as

$$a = \begin{pmatrix} \varepsilon_{2n} & 0 \\ 0 & \varepsilon_{2n}^{-1} \end{pmatrix}; \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here ε_{2n} is a primitive $2n$ th root of unity. We will identify BD_{4n} with its image under this representation. Thus when $A_1 = a, A_2 = b$ we have $x_1 = \cos(2\pi k/2n), x_2 = x_{12} = 0$. We call a, b standard generators of BD_{2n} .

We note that if n is odd, then the matrices $a' = \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix}, b' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ still generate a binary dihedral group with standard generators $a(b')^2, b'$.

Lemma 5.1. *(a) If $1 \leq i \leq 7$ and $p \in \mathcal{P}_3 \cap \mathcal{U}_i \cap V(F)$, and p corresponds to the matrices A_1, A_2, A_3 , then $\langle A_1, A_2, A_3 \rangle$ is a binary dihedral group.*

(b) If $\langle A_1, A_2, A_3 \rangle$ is a binary dihedral group, then the corresponding point $p \in \mathbb{R}^7$ is in $\mathcal{P}_3 \cap \mathcal{U}_i \cap V(F)$ for some $1 \leq i \leq 7$.

Proof Assume without loss of generality that $p \in \mathcal{U}_1 \cap V(F)$, so that we have $x_3 = x_{13} = x_{23} = x_{123} = 0$. Since $x_3 = 0$ we see that A_3 has characteristic polynomial $x^2 + 1$, so that A_3 has order 4 and is conjugate to b . We thus may assume that $A_3 = b$.

Since $p = (x_1, x_2, 0, x_{12}, 0, 0, 0) \in \partial\mathcal{T}_1 \cap P_3$ we see that $E(x_1, x_2, x_{12}) = 1$ so that by Lemma 3.1 we have $x_1 = \cos(2\pi u/v), x_2 = \cos(2\pi p/q)$. Let $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where we must have

$$a + d = 2\cos(2\pi u/v), \quad ad - bc = 1, \quad c - b = 0.$$

The last equation comes from the fact that $x_{13} = 0$. Solving these equations gives

$$a = 2\cos(2\pi u/v) - d, \quad b = c = \sqrt{-d^2 + 2d\cos(2\pi u/v) - 1}.$$

Now any matrix of the form $R = \begin{pmatrix} r & s \\ -s & r \end{pmatrix}, r, s \in \mathbb{C}$, commutes with $b = A_3$, and so we can replace A_1 by $R^{-1}A_1R$. One then solves the $(1, 2)$ entry of this matrix for s . This has the effect of making $R^{-1}A_1R$ a diagonal matrix with trace $2\cos(2\pi u/v)$. It follows that $R^{-1}A_1R = \begin{pmatrix} \exp 2\pi i u/v & 0 \\ 0 & \exp -2\pi i u/v \end{pmatrix}$. We assume that A_1 is this latter matrix.

Now let $A_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, so that we must have

$$e + h = 2\cos(2\pi p/q), \quad eh - fg = 1, \quad f = g,$$

where the last equation comes from the fact that $x_{23} = 0$.

Then the condition $x_{123} = 0$ gives $g = f \exp 4\pi i u/v$. This, with $f = g$, gives either (i) $f = g = 0$; or (ii) $v = 1, 2$. If we have (i), then A_2 is also diagonal of finite order and we are done. If we have (ii), then $x_1 = \pm 1$, and $A_1 = \pm I_2$; thus $\langle A_1, A_2, A_3 \rangle$ is a binary dihedral group. This proves (a), and (b) follows easily. \square

6. A REVIEW OF A PROOF FOR $n = 2$

We briefly review one of the proofs of Theorem 1 from [8]. In [8], for the case $n = 2$, the authors define

$$g = \begin{pmatrix} 2 & 2x_1 & 2x_{12} \\ 2x_1 & 2 & 2x_2 \\ 2x_{12} & 2x_2 & 2 \end{pmatrix},$$

showing that for (x_1, x_2, x_{12}) in the interior of \mathcal{T} the matrix g is positive definite. This uses Lemma 3.5 and the fact that

$$\det(g) = 8(1 - x_1^2 - x_2^2 - x_{12}^2 + 2x_1x_2x_{12}) = 8(1 - E(x_1, x_2, x_{12})).$$

This then enables them to define a reflection subgroup $\langle R_1, R_2, R_3 \rangle$. Here

$$R_1 = \begin{pmatrix} -1 & -2x_1 & -2x_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2x_1 & -1 & -2x_2 \\ 0 & 0 & 1 \end{pmatrix};$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2x_{12} & -2x_2 & -1 \end{pmatrix}.$$

These reflections have the property that

$$R_i^T g R_i = g, \text{ for } i = 1, 2, 3.$$

The authors of [8] then note that $\langle R_1, R_2, R_3 \rangle$ is a subgroup of the orthogonal group (of 3×3 real matrices) determined by the positive definite form g .

They also show that the x_1, x_2, x_{12} have the form $\cos(\pi p_u/q)$ for $p_u, q \in \mathbb{Z}, u = 1, 2, 12$. This enables them to define the above matrices over $\mathbb{Q}[\zeta_q + 1/\zeta_q] = \mathbb{Q}[\cos(\pi/q)]$. Using the Galois group \mathcal{G} of $\mathbb{Q}[\cos(\pi/q)]/\mathbb{Q}$ it is clear that for $\gamma \in \mathcal{G}$ each $\gamma(R_i)$ fixes the form $\gamma(g)$. Take the direct sum

$$\bigoplus(g) = \bigoplus_{\gamma \in \mathcal{G}} \gamma(g),$$

and consider the direct product group

$$\bigoplus(\langle R_1, R_2, R_3 \rangle) = \bigoplus_{\gamma \in \mathcal{G}} \gamma(\langle R_1, R_2, R_3 \rangle).$$

From the above it is clear that $\bigoplus(g)$ is a positive definite form on \mathbb{R}^{3N} , $N = |\mathcal{G}|$, fixed by the group $\bigoplus(\langle R_1, R_2, R_3 \rangle)$; further, as the group $\langle R_1, R_2, R_3 \rangle$ can be defined over the ring of integers \mathcal{O}_q of $\mathbb{Q}[\zeta_q]$, there is a lattice in \mathbb{R}^{3N} that is fixed by $\bigoplus(\langle R_1, R_2, R_3 \rangle)$ (namely \mathcal{O}_q^{3N}). Thus $\bigoplus(\langle R_1, R_2, R_3 \rangle)$ is a crystallographic group. Since $\bigoplus(\langle R_1, R_2, R_3 \rangle)$ is a discrete subgroup of the compact orthogonal group of \mathbb{R}^{3N} , $N = |\mathcal{G}|$, we see that $\bigoplus(\langle R_1, R_2, R_3 \rangle)$ is finite, and so $\langle R_1, R_2, R_3 \rangle$ is also finite.

This then limits the possibilities for $\langle R_1, R_2, R_3 \rangle$ to a finite number of cases, using Coxeter's classification of finite reflection groups [6]. For each case that has trace triples inside \mathcal{T} we obtain a finite group $\langle A_1, A_2 \rangle$. This gives the result when $n = 2$.

7. REFLECTION GROUPS FOR $n = 3$

Given any symmetric $n \times n$ real matrix $g = (g_{i,j})$ with 2s on the diagonal we can form reflections $R_1, \dots, R_n \in \text{GL}(n, \mathbb{R})$ where R_i is the identity, except that the i th row of R_i is

$$-g_{i,1}, -g_{i,2}, \dots, -g_{i,i-1}, -1, -g_{i,i+1}, \dots, -g_{i,n}.$$

We say that *the R_i are formed using g* . Then one can check that R_1, \dots, R_n fix g in the sense that $R_i^T g R_i = g$ for $i = 1, \dots, n$.

In general, given matrices $A_1, \dots, A_n \in \text{SL}(2, \mathbb{C})$ and letting $A_0 = I_2$, we can form the $(n+1) \times (n+1)$ symmetric matrix (with rows and columns indexed by $0, 1, \dots, n$)

$$g_n = \left(\frac{\text{trace}(A_i A_j^{-1})}{2} \right).$$

Thus for example, when $n = 2$, we get the matrix

$$g_2 = g_2(A_1, A_2) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 2x_1x_2 - x_{12} \\ x_2 & 2x_1x_2 - x_{12} & 1 \end{pmatrix}.$$

One checks that

$$\det(g_2) = 1 - x_1^2 - x_2^2 - x_{12}^2 + 2x_1x_2x_{12} = 1 - E(x_1, x_2, x_{12}),$$

and it easily follows that g_2 is positive definite on the interior of \mathcal{T} .

For $n = 3$, and A_1, A_2, A_3 associated to $p = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$, we have:

$$g_3 = g_3(p) = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & 1 & 2x_1x_2 - x_{12} & 2x_1x_3 - x_{13} \\ x_2 & 2x_1x_2 - x_{12} & 1 & 2x_2x_3 - x_{23} \\ x_3 & 2x_1x_3 - x_{13} & 2x_2x_3 - x_{23} & 1 \end{pmatrix}.$$

Our goal will be to show that g_3 is close to being positive-definite on $(-1, 1)^7$. Here

$$\begin{aligned} \det(g_3) = & 1 - x_{23}^2 - x_{13}^2 - x_3^2 - x_{12}^2 - x_2^2 - x_1^2 + 2x_{23}x_3x_2 + 2x_2x_1x_{12} \\ & + 2x_3x_1x_{13} + x_1^2x_{23}^2 + x_3^2x_{12}^2 + x_2^2x_{13}^2 + 4x_3^2x_2^2x_1^2 + 2x_2x_1x_{23}x_{13} \\ & + 2x_{12}x_2x_{13}x_3 - 4x_1^2x_{23}x_3x_2 - 4x_3^2x_{12}x_2x_1 - 4x_2^2x_3x_1x_{13} \\ & + 2x_1x_{23}x_3x_{12} - 2x_{12}x_{13}x_{23}. \end{aligned}$$

Lemma 7.1. *On $(-1, 1)^7 \cap V(F)$ the function $\det(g_3)$ is positive, except on a 5-dimensional subset of $V(F)$, where it is zero.*

Proof We just need to note that

$$\det(g_3) + F = (2x_1x_2x_3 + x_{123} - x_3x_{12} - x_1x_{23} - x_2x_{13})^2,$$

so that letting

$$G = 2x_1x_2x_3 + x_{123} - x_3x_{12} - x_1x_{23} - x_2x_{13},$$

we see that $V(F) \cap V(G)$ has dimension 5, and the result follows. \square

Remark For $n \geq 4$ the matrix g_n is an $(n+1) \times (n+1)$ matrix where $n+1 \geq 5$, and so by [15, Theorem 2] we see that $\det(g_n) = 0$. (The result of [15] says that if $m_1, \dots, m_k, M_1, \dots, M_k \in \text{SL}(2, \mathbb{C}), \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$, and $D = (\text{trace}(m_i M_j^{\varepsilon_i}))$ with $k \geq 5$, then $\det(D) = 0$.) Thus there is no analogue of Lemma 7.1 when $n \geq 4$, and so this method of proof will not work for $n \geq 4$.

Now given A_1, A_2, A_3 with $x_1 = \text{trace}(A_1)/2$ etc, and $g = g_3, g = (g_{ij})$, we define

$$\begin{aligned} R_1 = & \begin{pmatrix} -1 & -2g_{12} & -2g_{13} & -2g_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2g_{21} & -1 & -2g_{23} & -2g_{24} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \\ R_3 = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2g_{31} & -2g_{32} & -1 & -2g_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2g_{41} & -2g_{42} & -2g_{43} & -1 \end{pmatrix}. \end{aligned}$$

Thus R_1, R_2, R_3, R_4 are formed using $2g_3$, a real symmetric matrix with 2s on the diagonal. Thus g_3 is fixed by any element of the group $\langle R_1, R_2, R_3, R_4 \rangle$: for $R \in \langle R_1, R_2, R_3, R_4 \rangle$ we have $R^T g_3 R = g_3$.

Proposition 1. *Assume that $p \in V(F) \cap (-1, 1)^7 \cap \mathcal{P}_3, p \notin V(G)$. Let the associated matrices be A_1, A_2, A_3 . Then either*

- (i) $p \in \mathcal{U}_i$ for some $1 \leq i \leq 7$, or
- (i) $\langle A_1, A_2, A_3 \rangle$ is finite, or
- (iii) $p \in \text{Image}(\Pi_3)$, or
- (iv) the matrix g_3 is positive-definite at p .

Proof For positive-definiteness we just check the usual conditions (Sylvester's criterion) on the determinants of the various $k \times k$ principal submatrices of g_3 , $k = 1, 2, 3, 4$. The case $k = 1$ is clear.

When $k = 2$ the principal 2×2 sub-matrix is $\begin{pmatrix} 1 & x_1 \\ x_1 & 1 \end{pmatrix}$, which has determinant $1 - x_1^2 > 0$, since $p \in (-1, 1)^7$.

When $k = 3$ the principal 3×3 sub-matrix with rows and columns with indices $0, 1, 2$ has determinant $1 - E_1(x_1, x_2, x_{12})$. Now if $1 - E_1(x_1, x_2, x_{12}) < 0$, then $E(x_1, x_2, x_{12}) > 1$, and using Lemma 3.5 we see that there is a point p' in the $\text{Aut}(F_3)$ -orbit of p such that p' has a coordinate that is greater than 1 in absolute value. We can assume that $p' = (p_1, p_2, \dots, p_{123})$ where $|p_1| > 1$. But Lemma 3.2 shows that A_1 has finite order, so that $|p_1| \leq 1$, a contradiction.

On the other hand, if $1 - E_1(x_1, x_2, x_{12}) = 0$, then Lemma 3.4 tells us that (x_1, x_2, x_{12}) has an element in its $\text{Aut}(F_2)$ -orbit that has a 1 as one of its entries. It follows that p has an element in its $\text{Aut}(F_3)$ -orbit that has a 1 as one of its entries. Thus by Lemma 4.2 we see that either $p \in \mathcal{U}_i$, for some $1 \leq i \leq 7$, or $\langle A_1, A_2, A_3 \rangle$ is finite. This does the case $k = 3$.

For the $k = 4$ case we note that since $p \notin V(G)$, Lemma 7.1 gives $\det(g_3) > 0$. \square

The above shows that either we have what we want for Theorem 2 (namely that $p \in \text{Image}(\Pi_3)$), or p is on an axis, or $\langle A_1, A_2, A_3 \rangle$ is finite, or $p \in \mathcal{T}_i$, or $G(p) = 0$, or $g_3(p)$ is positive-definite.

Now we show how to deal with the fact that on the set $V(F) \cap V(G)$ the form g_3 is not positive definite.

We recall the *Cholesky decomposition*: the (real) matrix M is positive definite if and only if there exists a (real) unique lower triangular matrix L , with real and strictly positive diagonal elements, such that $M = LL^T$. For the matrix $M = g_3$ we find that $L = L(p)$ has the form

$$(7.1) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & \sqrt{1-x_1^2} & 0 & 0 \\ x_2 & \frac{x_1 x_2 - x_{12}}{\sqrt{1-x_1^2}} & \frac{\sqrt{(1-x_1^2)(1-E(x_1, x_2, x_{12}))}}{x_1^2 - 1} & 0 \\ x_3 & \frac{x_1 x_3 - x_{13}}{\sqrt{1-x_1^2}} & \frac{W}{\sqrt{(1-x_1^2)(1-E(x_1, x_2, x_{12}))}} & \frac{\sqrt{(1-E(x_1, x_2, x_{12})) \det(g_3)}}{E(x_1, x_2, x_{12}) - 1} \end{pmatrix},$$

where $W = -x_1 x_2 - x_1 x_3 x_{12} + x_{12} x_{13} - x_1 x_2 x_{13} + 2x_1^2 x_2 x_3 - x_1^2 x_{23} + x_{23}$.

We note that each element whose square root is taken is non-negative, since we can assume that $x_1^2 < 1$, $E(x_1, x_2, x_{12}) < 1$ and $\det(g_3) \geq 0$. This again reduces the question of whether g_3 is positive definite to the question of whether G is non-zero.

Now we see what happens if we replace A_1, A_2, A_3 by $\alpha(A_1), \alpha(A_2), \alpha(A_3)$, $\alpha \in \text{Aut}(F_3)$. Suppose that

$$(p)\alpha = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\alpha = (y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}).$$

Then $L((p)\alpha)$ is given by (7.1) with y_I replacing x_I . Of course $\langle A_1, A_2, A_3 \rangle = \langle \alpha(A_1), \alpha(A_2), \alpha(A_3) \rangle$. Further, the positive-definiteness of $\det g_3((p)\alpha)$ now depends on whether $G((p)\alpha) = G(y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123})$ is non-zero or not. If it is non-zero, then $g_3((p)\alpha)$ is positive-definite, which is what we want.

Thus the 'bad' case is if $G((p)\alpha)$ is zero for all $\alpha \in \text{Aut}(F_3)$. We thus look at the smallest ideal that contains G and F and is $\text{Aut}(F_3)$ -invariant. A Magma

computation shows that this ideal is \mathcal{X} . Thus in this (bad) case $p \in \mathcal{X}$, and we are done by Lemma 4.1. Thus we have part (a) of:

Proposition 2. *Assume that $p \in \mathcal{P}_3$, where $p \in V(F) \cap (-1, 1)^7$, p is not on a coordinate axis or in any of the ∂T_i , $1 \leq i \leq 7$, and $p \notin \text{Image}(\Pi_3)$. Let R_1, R_2, R_3, R_4 be the reflections formed from $2g_3(p)$. Then*

- (a) *the matrix $g_3(p)$ is positive-definite at p ;*
- (b) *$\langle R_1, R_2, R_3, R_4 \rangle$ is a finite reflection group.*

Proof Part (b) follows from (a) as in the proof of the $n = 2$ case outlined in §6. \square

Now to complete the proof of our result for $n = 3$ we need only consider the possibilities for R_1, R_2, R_3, R_4 and $G = \langle R_1, R_2, R_3, R_4 \rangle$ using Coxeter's classification of finite groups generated by reflections [6].

First note that each R_i is determined by a vector $v_i \in \mathbb{R}^4$ such that

$$R_i(v) = v - 2 \frac{(v, v_i)}{(v_i, v_i)} v_i, \quad 1 \leq i \leq 4, \quad v \in \mathbb{R}^4.$$

Here (\cdot, \cdot) is the form determined by g_3 . We further note that distinct reflections R_i, R_j with vectors v_i, v_j (respectively) commute if and only if $(v_i, v_j) = 0$.

Now from the theory of Coxeter groups [14] the decomposition of G as a direct sum is determined by the components of the graph $\Gamma(G)$ whose vertices are v_1, \dots, v_4 and where we have an edge v_i, v_j whenever $(v_i, v_j) \neq 0$. Thus any such finite reflection group G decomposes as a direct product of irreducible reflection groups, one for each component of $\Gamma(G)$.

Coxeter's classification [6] shows that there are five of these groups that are irreducible.

If G is a direct product of two irreducible finite reflection groups, then we have either

- (ai) $G = G_3 \times G_1$; or (aii) $G = G_2 \times G'_2$.

Here G_i, G'_i are irreducible finite reflection groups of degree i , and they are generated by i of the reflections R_1, R_2, R_3, R_4 . Clearly, there is only one such group of degree 1. Similarly, if G is a direct product of three irreducible finite reflection groups, then we have

- (bi) $G = G_2 \times G_1 \times G_1$.

Lastly, if G is a direct product of four irreducible finite reflection groups, then we have

- (ci) $G = G_1 \times G_1 \times G_1 \times G_1$.

If G is irreducible, then G has type A_4, B_4, D_4, F_4, H_4 of orders 120, 384, 192, 1152, 14400 (respectively). In these cases the order of $R_i R_j$, $1 \leq i, j \leq 4$ is in $\{1, 2, 3, 4, 5\}$, so that the values of x_1, x_2, \dots, x_{123} are $\cos(\frac{\pi k}{m})$, where $m = 1, 2, 3, 4, 5$ and $0 \leq k \leq m$; see Corollary 1.1. There are eleven such cosine values, including ± 1 .

This reduces the checking of this (irreducible) case to a finite number of cases. We note that if one of x_1, x_2, \dots, x_{123} is ± 1 , then this situation is covered by Lemma 4.2. Thus there are now nine possible values for x_1, x_2, \dots, x_{123} (the $\cos(\frac{\pi k}{m})$, where $m = 1, 2, 3, 4, 5$ and $0 \leq i \leq m$, that are not ± 1).

Checking (using [2]) we find that the only situation where the $\text{Aut}(F_3)$ orbit of $(x_1, x_2, \dots, x_{123})$ is finite is when the orbit has size one of 168, 3360, 336, 520, 112, with the corresponding group G having order 32, 48, 100, 192, 36 (respectively).

However one checks that each such orbit has a point in it that contains a 1, so that these cases are also covered by Lemma 4.2. (We note that the groups G so found are not necessarily irreducible, but this does cover all such irreducible G .)

Now if we have a group G of type (ai), then the possibilities for G_3 are the Coxeter groups of type A_3, B_3, H_3 of orders 24, 48, 120 (respectively). The possibilities for the values of x_1, x_2, \dots, x_{123} are the same as in the irreducible case just considered, and so these cases are covered by the calculations for the irreducible case above.

Now, in considering cases (aii) and (bi), the possibilities for G_2 are the Coxeter groups of type $I_2(k)$ i.e. the dihedral groups D_{2k} of order $2k$.

Thus if we have (aii) or (bi) with $\langle R_i, R_j \rangle \cong D_{2u}, u \geq 3$, then $G = \langle R_i, R_j \rangle \times H$, where H is generated by the $R_k, k \neq i, j$. Thus we have $(v_i, v_k) = (v_j, v_k) = 0$ for $k \neq i, j$. This shows that four of the $x_1, x_2, x_3, 2x_1x_2 - x_{12}, 2x_1x_3 - x_{13}, 2x_2x_3 - x_{23}$ are zero.

For example, suppose that $i = 1, j = 2$. Then we see that

$$x_2 = x_3 = 2x_1x_2 - x_{12} = 2x_1x_3 - x_{13} = 0,$$

which gives $x_2 = x_3 = x_{12} = x_{13} = 0$. This shows that $p = (x_1, 0, 0, 0, 0, x_{23}, x_{123}) \in \mathcal{U}_4$. Similarly, if $i = 1, j = 3$, then we get $x_1 = x_3 = x_{12} = x_{23} = 0$, which gives $p = (0, x_2, 0, 0, x_{13}, 0, x_{123}) \in \mathcal{U}_5$. If $i = 1, j = 4$, then we get $x_1 = x_2 = x_{13} = x_{23} = 0$, giving $p = (0, 0, x_3, x_{12}, 0, 0, x_{123}) \in \mathcal{U}_6$. The rest of the cases are similar. This concludes consideration of (aii) and (bi).

The last case, (ci), is not allowed, since $p \notin V(F)$ in this case. This concludes the proof of Theorem 2. \square

We note that in Theorem 2, any $p \in \mathcal{P}_3 \cap V(F)$ of type (i) has its only non-zero entry equal to ± 1 (since $p \in V(F)$), and so is in some $\partial \mathcal{T}_i, 1 \leq i \leq 7$; it is thus covered by type (ii). Further, from the results of §5, we see that any $p \in \mathcal{P}_3$ of type (ii) has $\langle A_1, A_2, A_3 \rangle$ a binary dihedral group, and so is covered by type (iii). Thus we now have

Corollary 7.2. *If $p \in \mathcal{P}_3 \cap V(F)$, then $p \in \mathcal{F}_3$, and we have one of the following:*

(i) $p \in \text{Image}(\Pi_3)$;

(ii) *the associated group $\langle A_1, A_2, A_3 \rangle$ is finite.* \square

8. THE PROOF OF THE $n > 3$ CASES

In general we have the polynomial ring $R_n = \mathbb{Q}[x_1, x_2, \dots, x_{12\dots n}]$. We let \mathfrak{T}_n be the *trace ideal* of R_n consisting of elements $x = x(x_1, x_2, \dots, x_{12\dots n}) \in R_n$ such that for all $A_1, A_2, \dots, A_n \in \text{SL}(2, \mathbb{R})$ we have

$$x(\text{trace}(A_1)/2, \text{trace}(A_2)/2, \dots, \text{trace}(A_1 A_2 \dots A_n)/2) = 0.$$

The *trace ring* is R_n/\mathfrak{T}_n , and there is a well-defined action of $\text{Aut}(F_n)$ on R_n/\mathfrak{T}_n . The ideal \mathfrak{T}_n determines a subset $V(\mathfrak{T}_n)$ of \mathbb{R}^{2^n-1} that is thus invariant under the action of $\text{Aut}(F_n)$, and on which there is an action of $\text{Aut}(F_n)$.

In Lemma 2.1 we have shown that there is a well-defined action of $\text{Aut}(F_3)$ on all of \mathbb{R}^7 given by the action on the traces; in the $n = 2$ case there was also a well-defined action of $\text{Aut}(F_2)$ on \mathbb{R}^3 . In order to obtain an action of $\text{Aut}(F_n), n > 3$, (also given by the action on the traces) we will have to restrict to the action on the points $(x_1, x_2, \dots, x_{123\dots n}) \in \mathbb{R}^{2^n-1}$ where there are $A_1, A_2, \dots, A_n \in \text{SL}(2, \mathbb{R})$ such that $x_1 = \text{trace}(A_1)/2, x_2 = \text{trace}(A_2)/2, \dots, x_{12\dots n} = \text{trace}(A_1 A_2 \dots A_n)/2$. The point here is that if we define the action of $\alpha \in \text{Aut}(F_n)$ on $\mathbb{Q}[x_1, x_2, \dots, x_{12\dots n}]^{2^n-1}$,

using the trace identities, then for $p = (x_1, x_2, \dots, x_{12\dots n}) \in \mathbb{Q}[x_1, x_2, \dots, x_{12\dots n}]^{2^n-1}$, with $(p)\alpha = (y_1, y_2, \dots, y_{12\dots n})$, we see that $x_I - y_I$ will be in the trace ideal \mathfrak{T}_n , meaning that $x_I(A_1, A_2, \dots, A_n) = y_I(A_1, A_2, \dots, A_n)$ for all such I .

Lemma 8.1. *If $p \in \mathcal{P}_3$ has finite associated matrix group, then the $\text{Aut}(F_3)$ orbit of p contains a point with an entry that is 1.*

Proof If the group $\langle A_1, A_2, A_3 \rangle$ is cyclic, then it is generated by one of $A_1, A_2, A_1A_2, A_1A_3, A_2A_3, A_1A_2A_3$. If it is generated by A_1 , then there is some $m \in \mathbb{N}$ such that $A_2A_1^m = I_2$, and we are done. The other cases are similar.

If the group $\langle A_1, A_2, A_3 \rangle$ is binary dihedral with standard generators a, b (as in §5), then one of the A_i has the form a^kb or a^kb^{-1} (for some k). By an automorphism we can assume that A_1 has the form a^kb . By a further automorphism we may assume that A_2, A_3 are each powers of a , so that $\langle A_2, A_3 \rangle$ is a cyclic group $\langle a^u \rangle$. Thus this case now follows as in the cyclic group case just considered.

Other than the cyclic and binary dihedral groups, there are three finite groups to consider: For BT_{24} (or a subgroup of BT_{24}) we find that there are 7 orbits of sizes 1, 7, 13, 14, 28, 91, 520. For BO_{48} (or a subgroup of BO_{48}) we find that there are 11 orbits of sizes 1, 7, 13, 14, 28, 91, 112, 168, 224, 520, 3360. For BI_{120} (or a subgroup of BI_{120}) we find that there are 12 orbits of sizes 1, 7, 13, 14, 28, 62, 91, 112, 336, 434, 520, 26688. In every case each orbit has an element that contains 1 as an entry. \square

Remark We gave the sizes of the orbits for subgroups of BI_{24} in the proof of the last result. One finds that the permutation representations of $\text{Aut}(F_3)$ that one obtains in each case have orders

$$1, \quad 2^3 \cdot 3 \cdot 7, \quad 2^4 \cdot 3^3 \cdot 13, \quad 2^{10} \cdot 3 \cdot 7, \quad 2^{11} \cdot 3 \cdot 7, \quad 2^7 \cdot 3^4 \cdot 7 \cdot 13, \\ 2^{190} \cdot 3^{17} \cdot 5^{13} \cdot 7 \cdot 13,$$

(respectively).

For the orbits of subgroups of BO_{48} one similarly obtains the following permutation group orders:

$$1, \quad 2^3 \cdot 3 \cdot 7, \quad 2^4 \cdot 3^3 \cdot 13, \quad 2^{10} \cdot 3 \cdot 7, \quad 2^{11} \cdot 3 \cdot 7, \quad 2^7 \cdot 3^4 \cdot 7 \cdot 13, \\ 2^{29} \cdot 3^8 \cdot 7, \quad 2^{10} \cdot 3 \cdot 7, \quad 2^{19} \cdot 3 \cdot 7, \quad 2^{190} \cdot 3^{17} \cdot 5^{13} \cdot 7 \cdot 13, \\ 2^{1037} \cdot 3^{64} \cdot 5^{28} \cdot 7^{29},$$

(respectively).

For the orbits of subgroups of BI_{120} one obtains the following permutation group orders:

$$1, \quad 2^3 \cdot 3 \cdot 7, \quad 2^4 \cdot 3^3 \cdot 13, \quad 2^{10} \cdot 3 \cdot 7, \quad 2^{11} \cdot 3 \cdot 7, \quad 2^2 \cdot 3 \cdot 5^3 \cdot 31, \quad 2^7 \cdot 3^4 \cdot 7 \cdot 13, \\ 2^{29} \cdot 3^8 \cdot 7, \quad 2^{29} \cdot 3^8 \cdot 5^7 \cdot 7, \quad 2^8 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31, \quad 2^{190} \cdot 3^{17} \cdot 5^{13} \cdot 7 \cdot 13, \\ 2^{6671} \cdot 3 \cdot 7 \cdot (1668!).$$

(respectively). These calculations were accomplished using Magma.

One says that the n -tuple A_1, A_2, \dots, A_n of elements of $\text{SL}(2, \mathbb{C})$ is *conjugate to an upper-triangular n -tuple* if there is $g \in \text{SL}(2, \mathbb{C})$ such that each of $A_1^g, A_2^g, \dots, A_n^g$ is an upper-triangular matrix.

Lemma 8.2. *Let $p \in \mathcal{P}_3$ correspond to the triple of matrices $A_1, A_2, A_3 \in \mathrm{SL}(2, \mathbb{C})$. Then the triple (A_1, A_2, A_3) is conjugate to a triple of upper-triangular matrices if and only if $p \in \mathrm{Image}(\Pi_3)$.*

Proof If (A_1, A_2, A_3) is conjugate to a triple of upper-triangular matrices, then we may assume that $A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & 1/a_2 \end{pmatrix}$, $A_3 = \begin{pmatrix} a_3 & b_3 \\ 0 & 1/a_3 \end{pmatrix}$. Since $p \in \mathcal{P}_3$ it follows that A_1, A_2, A_3 have finite order, or are $\pm K$, where K is an upper-triangular parabolic (see §4). Thus the eigenvalues $a_i, 1/a_i$ of A_i are $\zeta_i, 1/\zeta_i$, roots of unity. Thus there are $\theta_i \in \pi\mathbb{Q}$, $i = 1, 2, 3$, such that $x_i = \cos(\theta_i)$, $x_{ij} = \cos(\theta_i + \theta_j)$, $x_{ijk} = \cos(\theta_i + \theta_j + \theta_k)$, and it follows that $p \in \mathrm{Image}(\Pi_3)$.

The converse is given by [9, Corollary 2.11]. \square

Given A_1, A_2, \dots, A_n , $n \geq 3$ and $1 \leq i < j < k \leq n$, the triple A_i, A_j, A_k determines a 7-tuple of traces $(x_1, x_2, \dots, x_{123})$ on which we can evaluate F ; denote the resulting value by $F_{i,j,k}$. We immediately see that $F_{i,j,k} = 0$ for all such i, j, k . Thus we may apply Corollary 7.2 to the triple A_i, A_j, A_k .

For the main result ($n > 3$) we need to define $\Pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{2^n-1}$, $n > 3$, analogously to Π_2, Π_3 :

$$\begin{aligned} \Pi_n(t_1, t_2, \dots, t_n) = & (\cos(2\pi t_1), \cos(2\pi t_2), \dots, \cos(2\pi t_n), \cos(2\pi(t_1 + t_2)), \\ & \dots, \cos(2\pi(t_1 + t_2 + \dots t_n))). \end{aligned}$$

The proof of our main result proceeds by induction on $n \geq 3$, where Corollary 7.2 gives the $n = 3$ case. The result that we prove is

Theorem 3. *If $p \in \mathcal{P}_n$, $n \geq 3$, corresponds to the matrices A_1, \dots, A_n , then $p \in \mathcal{F}_n$, and we have one of the following:*

- (i) $p \in \mathrm{Image}(\Pi_n)$;
- (ii) the associated group $\langle A_1, A_2, \dots, A_n \rangle$ is finite.

In particular, $\mathcal{P}_n = \mathcal{F}_n$.

Proof So let $p \in \mathcal{P}_n$ correspond to the matrices A_1, \dots, A_n . If for each distinct triple $1 \leq i, j, k \leq n$ the group $\langle A_i, A_j, A_k \rangle$ is conjugate to an upper-triangular triple, then [9, Theorem 2.7] shows that the n -tuple A_1, A_2, \dots, A_n is conjugate to an upper-triangular n -tuple.

Now suppose that some triple of the matrices A_1, \dots, A_n is not conjugate to an upper-triangular n -tuple. By a permutation action of $\mathrm{Aut}(\mathcal{F}_n)$ (and for ease of notation) we may assume that this triple is A_1, A_2, A_3 . Then by Corollary 7.2 we see that $\langle A_1, A_2, A_3 \rangle$ is a finite group. Then by Lemma 8.1 we can assume that one of $x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}$ is 1. Thus there is some $\alpha \in \mathrm{Aut}(\mathcal{F}_3) \leq \mathrm{Aut}(\mathcal{F}_n)$ such that $(p)\alpha = (1, x'_2, x'_3, x'_{12}, x'_{13}, x'_{23}, x'_{123}, \dots)$, which implies that $A_1 = I_2$, since $\langle A_1, A_2, A_3 \rangle$ is finite. But with $A_1 = I_2$, the result now follows by induction. \square

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